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THE JOURNAL

OF THE

INDIAN MATHEMATICAL SOCIETY

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WITH THE CO-OPERATION OF

Dr. R. P. PARANJPYE, M.A., D.Sc.,
Prof. A. C. L. WILKINSON, M.A., F.R.A.S.,
and others.

Volume XV.

1923-1924.

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	For	*Substitute*
p. 183 line 4	G. T. Taylor	G. I. Taylor
p. 183 ,, 25	,,	,,
p. 184 ,, 2	showed	shows
p. 184 ,, 22	axis	axis of x
p. 184 ,, 25	A_2	$-A_2$
p. 184 ,, 31	Jeffrey	Jeffreys
p. 185 ,, 21	Jeffrey's	Jeffreys'
p. 185 ,, 23	which	the solution of which
p. 186 ,, 20	speed	shear
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Errata—II.

Page *135*, lines 4 and 5 of Question 1327

for	$\tan^{-1} \operatorname{sc} u \ \operatorname{dn} v.$	*read*	$\tan^{-1} (\operatorname{sc} u . \operatorname{dn} v)$
,,	$\tan^{-1} \operatorname{sc} v . \operatorname{dn} u.$	,,	$\tan^{-1} (\operatorname{sc} v . \operatorname{dn} u)$
,,	$\tan^{-1} k \operatorname{sd} u . \operatorname{cn} v.$	,,	$\tan^{-1} (k \operatorname{sd} u . \operatorname{cn} v)$
,,	$\tan^{-1} k \operatorname{sd} v . \operatorname{cn} u$	,,	$\tan^{-1} (k \operatorname{sd} v . \operatorname{cn} u)$

Page *192*, read Question 1363 as follows :—

$$(1)\ \operatorname{cs}\frac{K}{n}\ \operatorname{cs}\frac{2K}{n}\ \operatorname{cs}\frac{3K}{n} \ldots\ldots \operatorname{cs}\frac{(n-1)K}{n} = (1-k^2)^{\frac{n-1}{4}}$$

$$(2)\ \operatorname{ds}\frac{K+iK'}{n} . \operatorname{ds}\frac{2}{n}(K+iK') . \operatorname{ds}\frac{3}{n}(K+iK') \ldots . \operatorname{ds}\frac{n-1}{n}(K+iK')$$

$$= k^{n-1} . \{\operatorname{cn}(K+iK')\}^{\frac{n-1}{2}}$$

The Indian Mathematical Society.

PROGRESS REPORTS 1923—1924.

I.

The following gentlemen have been enrolled as Honorary Members of the Society :—

(1) Prof. H. F. Baker of Cambridge.
(2) Prof. E. T. Whittaker of Edinburgh.
(3) Prof. A. N. Whitehead of London.

2. The following gentlemen have been admitted as Ordinary Members of the Society :—

(1) P. P. Anantanarayana Aiyar, M.A., Professor of Mathematics, Judson College, Rangoon.
(2) Balder Puri, M.A. (Cantab.) of Punjab.

3. The Calcutta Mathematical Society appears to be somewhat indifferent as regards holding a Mathematical Conference at Calcutta and the Managing Committee of our Society has decided to hold the Fourth Mathematical Conference at Poona in March next. Details of the arrangements will be announced in the next Progress Report. Meanwhile the members are hereby requested to prepare papers to be read before the Conference and send them with abstracts to the undersigned before the 31st January 1924.

P. V. SESHU AIYAR,
Hon. Secretary.

31st August 1923.

II.

1. The Managing Committee of the Society has been reconstituted as follows :—

Balakram, Esq., M.A., I.C.S. (*President*).
S. Narayana Aiyar, Esq., M.A. (*Treasurer*).
V. B. Naik, Esq., M.A (*Librarian*).
M. T. Naraniengar, Esq., M.A.
P. V. Seshu Aiyar, Esq., B.A., L.T. } (*Secretaries*).
D. D. Kapadia, Esq., M.A , B Sc.
E. B. Ross, Esq., M.A.
Pandit Hemraj, M.A.
K. Ananda Rao, Esq., M.A.
S. V. Ramamurthi, Esq., M.A , I.C.S.
N. M. Shah, Esq., B.A.
Rev. A. Steichen, Ph.D., S J.

2. Mr N. Raghunatha Aiyangar. M.A., having resigned the Assistant Secretaryship of the Society, Mr. G. A. Srinivasan, M.A., L.T., (Assistant Professor, Presidency College, Madras) who was till now an Associate, has been appointed Assistant Secretary. Mr. S. A. Mani, M.A., L T. (Assistant Lecturer, Government College, Kumbakonam) has been appointed an Associate in the place of Mr. G. A. Srinivasan.

3. The following gentlemen have been admitted as members of the Society :—

(1) Dhirendra Kumar Sen, Esq., M.Sc., Professor, Rajaram College, Kolhapur.
(2) V. Gourisankaran, Esq., B.A., (Hons.), L.T., Asst. Professor, Presidency College, Madras.
(3) Hans Raj Gupta, Esq., B.A., Jand Street, Patiala.
(4) T. B. Hardikar, Esq, M.A., Professor of Mathematics, New Poona College, Poona.
(5) V. N. Kudva, Esq., B.A., I. C. S., Assistant Collector, Chittoor.
(6) B. V. Krishna Aiyangar, Esq., B.Sc., Science Assistant, Wesleyan Collegiate High School, Mysore.
(7) M. Lakshmanamurthi, Esq., M.A., Assistant Professor, Presidency College, Madras.
(8) N. Mahadeva Aiyar, Esq., B.A , I.C.S., Assistant Collector and Magistrate, Bankura.
(9) Dhanvantrai M. Mehta, Esq., M.A , Office of the Director of Public Instruction, Bhavnagar.
(10) A. C. Mukerji, Esq., M.A., Assistant Professor of Mathematics, Baroda College, Baroda.
(11) P. Natesan, Esq., Mathematics Lecturer, Collegiate High School, Tumkur.
(12) V. N. Patwari, Esq., B A. (Hons). Senior Professor of Mathematics, City College, Hyderabad (Deccan).
(13) G. Ramachandra Aiyar, Esq., M A., L T., Assistant Professor, Presidency College, Madras
(14) C. E. Vanhorn, Esq , M.A., Ph.D., Senior Lecturer in Mathematics, Judson College, Rangoon.
(15) B. N. Vaidya, Esq , King George English School, Dadar, Bombay.

4. Since the Journal has now been divided into two parts, one for articles of an advanced nature and the other for short notes and questions, and since the issues of each part for a year when separately bound will

SRINAGAR

be too small for one volume, it has been decided to have each volume to run over two years and to consist of twelve issues. Thus the issue for February 1924 is marked "Volume XV, No 7," and the paging in each part is continued from the previous issue.

5. The Statement of Accounts for the year ending 31st December 1923 is given overleaf.

6. The Fourth Mathematical Conference of the Society is to be held at Poona, as already announced, from the 15th to 18th of April, and the detailed programme for the same has been sent already.

P. V. SESHU AIYAR,
Honorary Joint Secretary.

10*th April* 1924.

III.

It is with great regret that the Managing Committee of the Society learnt of the sudden death of SIR ASUTOSH MUKERJI, an Honorary Member of the Society. The Committee has passed the following resolution :—

"The Managing Committee of the Indian Mathematical
"Society places on record its deep sense of sorrow
"at the death of Sir Asutosh Mukerji and offers
"its heartfelt condolences to the members of the
"bereaved family."

2 The Committee is glad to be able to report that the FOURTH MATHEMATICAL CONFERENCE held at POONA in April last was a great success. The success was mainly due to the exertions of the Reception Committee in general and of its President, Dr. R. P. Paranjype, and its Secretaries in particular. The Committee takes this opportunity of placing on record its best thanks to the Members and Office-bearers of the Reception Committee. It is believed that the members will have received the official report of the Conference.

3. In order to make our Society better known to the mathematical world, the following institutions have been added, at the suggestion of our President, to our "Presentation" List:—

(1) Academie des Sciences, Paris.

(2) Cambridge Philosophical Society, Cambridge.

(3) Carnegie Institute of Washington (U. S. A.)

(4) "Nature," c/o Messrs. Macmillan & Co., Ltd., St. Martin's Street, London, W. C. 2.

(5) "Science Progress," 50-A, Albermarle Street, London, W.1.

(6) The Director and Secretary, Science Museum, South Kensington, London, S.W. 7.

4. At the business meeting held at the time of the Conference, some resolutions regarding the system of circulation of the Journals received in the Library, the concession rate of subscription, and presentation of reprints of contributions to authors were passed, and these are receiving the consideration of the Managing Committee. The final decisions arrived at will be reported in the next Progress Report.

5. The following gentlemen have been admitted as members of the Society :—

(1) Father Enrique de Rafael, S. J., D.Sc., Professor of Mathematics, St. Xavier's College, Fort, Bombay.

(2) K. Chand, Esq., B.A. (Cantab.), F.R.A.S., Professor of Mathematics, Osmania University College, Hyderabad (Deccan).

(3) K.R. Gunjikar, Esq , B.Sc., M.A., Professor of Mathematics, Elphinstone College, Fort, Bombay.

(4) V. Tiruvenkatachari, Esq., M.A , Assistant, Government Training School, Guntur (at concession rate).

(5) D. A. Dandekar, Esq , B.A., 9, Essak Buildings, Naigaum Road, Dadar, Bombay (at concession rate).

(6) P. R. Vaidya, Esq., B.A., L C.E., Sub-Divisional Officer, P.W.D , Sahapur, Dt. Thana.

(7) Abdur Rahiman Gausekhan Gokak, Esq., B.A., Jamkhandi (behind Jame-Masjid), *via* Kudchi, M. & S. M. Ry.

P. V. SESHU AIYAR,
Hon. Joint Secretary.

17*th August* 1924.

THE INDAIN MATHEMATICAL SOCIETY.

Statement of Accounts for the Year 1922.

Receipts:—		Re.	A.	P.	**Expenditure:—**		Rs.	A.	P.
Balance from 1921 ...	...	7,978	4	5	Ordinary working expenses ...		323	12	0
Subscription from Members	...	679	14	0	Books and Periodicals ...	...	70	11	0
Life Member's subscription	...	150	0	0	Printing Journal ...	...	823	14	0
Subscriptions for Journal	...	187	15	0	Library Expenses ...	...	389	4	3
Interest on Investments...	...	336	14	5	Closing Balance ..	...	7,752	6	7
Miscellaneous ...	...	27	0	0					
		9,359	15	10			9,359	15	10

MADRAS,

October 1923.

S. NARAYANA AIYAR,

Hon. Treasurer.

THE INDIAN MATHEMATICAL SOCIETY.

Statement of Accounts for the Year 1923.

Receipts:—		Rs.	A.	P.	**Expenditure:—**		Rs.	A.	P.
Balance from 1922 ...	...	7,752	6	7	Ordinary Working Expenses ...	...	357	3	9
Subscription from Members	...	3,130	4	0	Books and Periodicals ...	...	280	15	9
Subscription for Journal	...	394	10	6	Printing Journal ...	...	784	4	0
Interest on Investments	...	311	8	10	Library Expenses ...	...	282	0	0
Miscellaneous ...	...	109	5	0	Closing Balance ...	...	9,993	11	5
		11,698	2	11			11,698	2	11

MADRAS,

1st April 1924.

S. NARAYANA AIYAR,

Hon. Treasurer.

14

A Statistical Study of Some Examination Marks—II*

By P. V. Seshu Ayyar and S. R. Ranganathan.

In the first paper † on this subject read before the Madras Session of the Indian Science Congress, it was stated "The average values of the correlation co-efficients of the various subjects with English may be taken to give a quantitative measure of the extent to which the power of expression in good English is necessary in the various subjects. These measures will be more reliable, if the correlation between the marks in the various subjects and the marks in English composition alone are found. It will be of great pedagogic interest to have those measures calculated."

This second paper has been prepared as a supplement on those lines. Marks in the several English papers were separately available only for two years, *viz.*, the 12th and the 13th years, reckoned from the first of the six years considered in the first paper. The correlation co-efficients for these two years have been calculated. It may be pointed out that for the examination considered there are four papers in English for any year—two on text-books prescribed for detailed study, one on composition from books set for non-detailed study, and the last on general composition. For convenience of reference, we have designated all the four papers taken together, by the general name 'English' and the last two papers taken together, by the name 'Composition.' In testing the power of expression, the last two papers are in a sense complementary. For in the third paper, the matter is previously given to the candidates in a crude form in their text-books and they have merely to reproduce the same and marshal details in a suitable manner to produce a desired effect. Whereas, the paper on general composition tests the candidate's general knowledge and his ability to express his ideas in correct English. The following table gives the correlation co-efficients between the various subjects and English and English Composition separately considered :

---

* Read before the Eleventh Indian Science Congress, Bangalore, 1924.

† *Vide* J. I. M. S., Vol. XIV, pp. 43—55.

Correlation Co-efficients between

		Maths.	Physics.	Chem.	Mod. Hist.
12th year	English.	·44	·38	·46	·54
	Composition.	·36	·34	·38	·42
13th year	English.	·45	·46	·59	·41
	Composition.	·43	·34	·22	·33

As was expected the correlation co-efficient is slightly different if composition alone is taken into account instead of the whole of English. But no definite conclusion can be drawn from these meagre data. It is desirable to calculate the co-efficient for a longer period of years, say ten years or so, as material becomes available, and take the average value as a suitable measure of the part played by the power of expression in answering the papers in various subjects. Though nothing definite can be said of the correlation between English and the various subjects from these data, yet the bye-products obtained in this analysis, *viz.*, the *mean mark* and the *standard deviation* in the various subjects, seem to call for some interesting remarks.

A comparison of the mean mark and of the standard deviation in the various subjects for the two years with their average values for the first six years considered in the first paper,* discloses certain noteworthy features. The first row contains the average values of these quantities for the six years considered in the first paper.

Mean.

	English.	Maths.	Physics.	Chem.	Modern History.
Average for six years.	39·3	41·5	40·3	39·1	38·0
12th year.	37·13	44·28	39·36	33·52	37·25
13th year	34·82	40·57	35·76	38·54	35·72

* l. c., p. 47.

Standard Deviation.

	English.	Maths.	Physics.	Chem.	Modern History.
Average for six years.	9·26	13·23	11·98	12·50	8·60
12th year.	6·48	19·26	16·50	13·80	8·97
13th year.	5·14	18·48	13·29	10·49	8·24

Recalling to memory Fig. II* of the first paper one may say that there is nothing unusual about the mean in the various subjects. Perhaps one remark may be made regarding the mean in English. Till the end of the 12th year the minimum for a pass in English was 40 and the mean was oscillating about this figure and in the first six years considered the lowest value reached by it was only 36·3. In the 13th year, the minimum was reduced to 35 and the mean had dropped so low as 34·82. It would be of interest to watch the progress of the mean in English for a few more years. It will throw much light on the interaction between the mean in any subject and the minimum for a pass in that subject, as indicated in our first paper. It has to be remembered that the average mean in the other subjects was about 5 marks more than the minimum and that in English alone it was slightly lower than the minimum. Now that the minimum was lowered to 35 in the 13th year, the average also has fallen down. It is not proper to draw any conclusions from the experience of a single year. Observation must be continued over a few more years. If the phenomenon that is observed in the 13th year continues to exist in other years also, reduction of the minimum cannot be said to produce the desired result and under such circumstances the only proper method for fixing the minimum for a pass would be not to define it to be an absolute percentage but as the mark that is at a prescribed distance from the mean, as suggested in our first paper.

While there is nothing markedly abnormal about the behaviour of the mean in the two years considered, it is just the other way with respect to the standard deviation. It must be remembered that the standard deviation would ordinarily be characterised by comparative steadiness. This is also corroborated by Fig. III† of our first paper. But, in English, Mathematics and Physics, the fluctuation in the standard deviation is far from ordinary. It is only just noticeable in Physics; but it is very marked in Mathematics and in English.

---

* l. c., p. 48. † l. c., p. 50.

In Mathematics, it has gone up considerably, whereas in English it has dwindled down very much; this means that there is a tendency to crowd the candidates about the mean to a greater extent than before in English and to disperse the candidates more evenly than before over the entire range of marks in the case of Mathematics. The question is : what inferences can we draw from this, or on what hypothesis can we explain this?

To take up the case of Mathematics first, much will depend upon whether the frequency curve has undergone any change in shape or type. It is found that it has not undergone any marked change ; it continues to be just of the same type as that shown in Fig. I of our first paper and continues to have small skewness. Again, much will also depend upon any variation in the standard of the examination, as determined from other independent considerations. Now, there is a widespread feeling among those competent to judge that the standard of the Mathematics papers has appreciably fallen during the last few of the years considered. Let us take this as an assumption and let us put it alongside the tremendous rise in standard deviation, the constancy of the skewness of the curve and the normality of the distance of the mean from the minimum.

These considerations lead us to infer, in the first place, that the reduction of the standard of the question paper has affected different types of candidates differently. In the first place, the constancy of the skewness and of the type of the frequency curve indicates that there has been no appreciable shift in the frequency from below the minimum to above the minimum. Secondly, the rise in standard deviation shows that the candidates above the average show a tendency to go nearer to the maximum and that those below the average, a tendency to go nearer to zero. The former of these phenomena has, associated with it, the disadvantage that the results of such an examination do not give sufficient help in picking out the best candidates. This is corroborated by the experience of the professors who have to select candidates for the Honours course on the basis of these marks. For, it is found that, not infrequently, getting even 75 or 80 p. c. in these examinations does not ensure a decided aptitude for the subject.

Let us then pass on to candidates who are below the average. It was remarked that they are found to show a tendency to slide down and go nearer to zero. This indicates that the reduction of the standard is not availed of profitably by the very class of students for whom it has probably been intended. On the other hand, reduction in the standard of the paper seems to lead to a tendency to lower the aim itself of the

teacher and of the taught and thc effect of this tendency is seen most in the case of those who are below the average. These considerations indicate that there is hardly anything gained by the lowering of the standard in Mathematics and that perhaps there is something lost.

The marked lowering of the standard deviation in English is very difficult to explain. Of course, in a subject like English, the standard deviation is bound to be lower than in other subjects. But, in the two years considered, it is much lower than what it was in the first six years. In fact, practically, about 80 % of the candidates came within the range 30 to 40 and nearly 50 % within the range $32\frac{1}{2}$ to $37\frac{1}{2}$. This indicates either that mediocrity is the order of the day in English, or that the examiners concentrate their attention more on whether to pass or fail a candidate than on finding a closer approximation to his actual worth.

This statistical analysis, combined with what is contained in our first paper, clearly proves the desirability of conducting annual statistical audits of the working of large-scale examination systems with a view to find out general tendencies in candidates and examiners and to enable educationists to recognise defects, if there are any, and to devise measures to remove them. The authorities seem to think, however, that, once the papers are set and valued and the results announced, their responsibility ends and that the examination system is functioning properly. But, as a matter of fact, our examination system is very artificial and unreliable, if conducted without proper safeguards and proper vigilance, as it obtains now. It makes us drift, we don't know in what direction, without our even realising that we are so drifting.

Hence, we once again impress on the public the imperative necessity for the appointment of an Examination Board (having a skilled statistician on it) connected with every University or other examining body—which in the words of the Calcutta University Commission should act as the "conscience" of the authorities in respect of examinations.

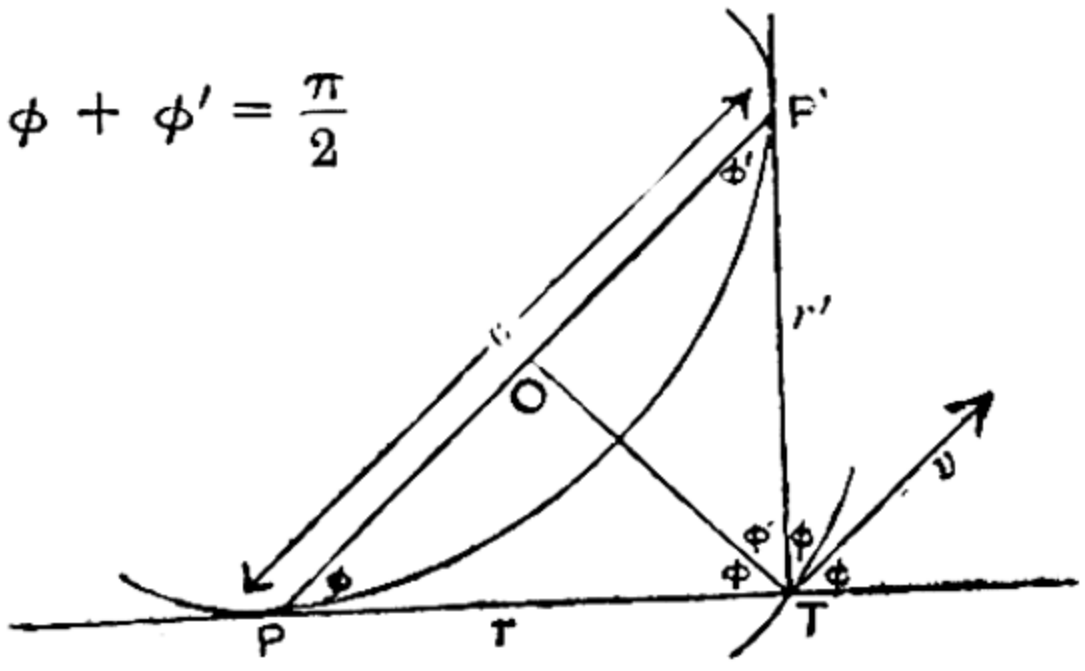
The Curvature of the Orthoptic Locus.

By A. Narasinga Rao, M.A.,
College of Engineering, Guindy.

[The orthoptic of a plane curve C is the locus of the intersection of perpendicular tangents to it. Taking such a pair of tangents as a set of moving axes, I obtain expressions (6) and (8) for the curvature of the orthoptic. From these are deduced certain characteristic properties of curves whose orthoptics are straight lines. Besides the parabola, we have such a curve in the unicursal quartic for which the line at infinity is a bitangent in two perpendicular directions.* Hence there arise between these two curves and others for which a straight line is part of the orthoptic locus certain points of similarity which, I think, have not been noticed hitherto. In the last paragraph, I prove that the orthoptic of the quartic referred to above is its directrix.]

1. Let the tangents at P and P′ meet at right angles at T, so that T traces the orthoptic locus. Also let s and s' be the arcual distances of P and P′ from a fixed point on the curve; r, r' the lengths of the tangents PT, P′T and ψ the angle which the tangent PT makes with a fixed line in the plane.

$$\phi + \phi' = \frac{\pi}{2}$$



Let the points P and P′ move along the curve dynamically; we have

vel. of T = vel. of P + vel. of T relative to P

$= (\dot{s} + \dot{r})$ along PT; and $(r\dot{\psi})$ along TP′. ... (1)

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* Hilton: *Plane Algebraic Curves*—p. 174., Ex. 13. Hilton evidently intends this to be proved by the use of Plucker's numbers for the orthoptic. The demonstration in § 3 is direct and dispenses with the use of these.

Again

vel. of T = vel. of P′ + vel. of T relative to P′

$= r'\dot{\psi}$ along PT; and $(\dot{s}' - \dot{r}')$ along TP′.... (2)

Equating the values derived from (1) and (2), we obtain

$$\left.\begin{aligned} \dot{s} + \dot{r} = r'\dot{\psi} \quad , \quad r\dot{\psi} = \dot{s}' - \dot{r}' ; \\ \text{i.e.} \qquad \rho + \frac{dr}{d\psi} = r' \text{ and } \rho' - \frac{dr'}{d\psi} = r ; \end{aligned}\right\} \quad \ldots \quad (3)$$

ρ and ρ' being the radii of curvature at P and P′ respectively.

Let v be the velocity of T along the orthoptic and $\bar{K}$ the curvature of the orthoptic at T, so that the angular velocity of the tangent at T is $\bar{K}v$.

We have then from the figure*

$$v \sin\phi = \dot{r} + \dot{s}, \quad \ldots \quad \ldots \quad (4)$$

$$\frac{v\cos\phi}{r} - \dot{\phi} = \bar{K}v ;$$

i.e.

$$\bar{K} = \frac{\cos\phi}{r} - \frac{\dot{\phi}}{v}. \quad \ldots \quad \ldots \quad (5)$$

Substitute for v in (5) the value

$$v = \frac{\dot{r} + \dot{s}}{\sin\phi} = \frac{r'\dot{\psi}}{\sin\phi} = c\dot{\psi}$$

derived from (4) and (3). We have thus

$$\bar{K} = \frac{\cos\phi}{r} - \frac{1}{c}\frac{d\phi}{d\psi}.$$

$$\therefore \quad \bar{K}c = 1 - \frac{d\phi}{d\psi}. \quad \ldots \quad \ldots \quad (6)$$

Again since $\cot\phi = \frac{r}{r'}$,

$$-r'^2 \operatorname{cosec}^2\phi \frac{d\phi}{d\psi} = r'\frac{dr}{d\psi} - r\frac{dr'}{d\psi} ;$$

i.e.

$$c^2\frac{d\phi}{d\psi} = r'(\rho - r') - r(r - \rho') \quad \text{from (3)}$$

$$= r\rho' + r'\rho - c^2. \quad \ldots \quad \ldots \quad (7)$$

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* The normal to the orthoptic locus at T bisects the ortho-chord PP′. (Hilton :—*Plane Algebraic Curves*, p 170) Hence the tangent makes an angle ϕ with PT.

Hence we obtain for the curvature of the orthoptic

$$\left.\begin{aligned}\bar{K} &= \frac{2c^2 - \rho_1 r' - \rho' r}{c^3} \\ &= \frac{2c - \rho \sin\phi - \rho' \sin\phi'}{c^2}.\end{aligned}\right\} \quad \ldots \qquad \ldots \quad (8)$$

Corollaries :

i. If P or P′ be a point of inflexion the corresponding point T is a cusp on the orthoptic locus.

ii. If P and P′ are cusps, $\rho = 0$ $\rho' = 0$ and $\bar{K} = \frac{2}{c}$. Hence the circle on PP′ as diameter is the circle of curvature at T.

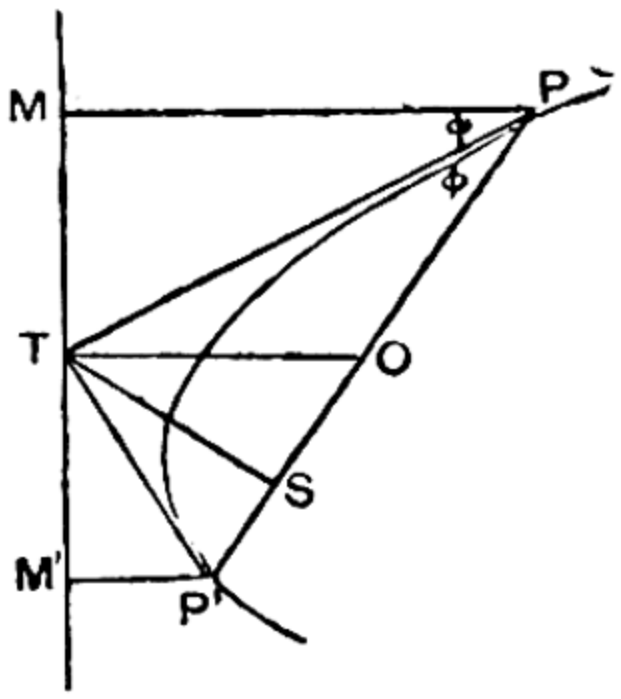
iii. T will be a point of inflexion if $2c^2 = \rho r' + \rho' r$. Hence inflexions may exist on the orthoptic at points corresponding to ordinary points on C.

2. If the orthoptic locus be a straight line, $\bar{K}$ vanishes at all points and hence by (6)

$$d\phi = d\psi.$$
$$\therefore \quad \phi = \psi,$$

if the direction $\psi = 0$ be properly chosen.

This result may also be obtained from the figure.



The normal to the orthoptic straight line MTM′ passes through O the middle point of PP′. Hence *TO is fixed in direction* and may be taken as the initial direction from which ψ is measured. If perpendiculars PM and P′M′ are drawn from P and P′ to the orthoptic line, we have

$$\angle \mathrm{MPT} = \text{alternate } \angle \mathrm{PTO}\ (= \psi)$$
$$= \angle \mathrm{TPO}\ (= \phi)$$

since OT = OP = OP′.

The tangent PT bisects the angle between a fixed direction PM and the ortho-chord through P. ... ... (9)

Again from the figure $\mathrm{PM} = c\cos^2\phi$, $\mathrm{P'M'} = c\sin^2\phi$;

and $$\mathrm{PM} + \mathrm{P'M'} = \mathrm{PP'}.$$

Hence

There exists a point S on PP′ such that SP = PM and SP′ = P′M.

This point is the projection of T on PP′. ... ... (10)

Lastly to each point P corresponds a single tangent PT meeting the orthoptic straight line in the unique point T and through this point passes

a unique tangent in a perpendicular direction touching the curve at P′. Also the relation between P and P′ is symmetric. Hence

The points P and P′ are corresponding members of an involution range on the curve. ... ... (11)

These properties follow from a consideration of the curve C in the immediate neighbourhood of P and P′. Hence they will hold even if

(i) The straight line is only a part of the orthoptic locus,

and (ii) P and P′ lie on different algebraic curves provided the tangents at the points meet at right angles on a straight line.

3. There are two cases of the orthoptic locus being a straight line, namely, when C is a parabola and when it is a curve of class 3 for which the line at infinity is a bi-tangent in two perpendicular directions. We have thus a striking resemblance between these two curves in (9), (10) and (11).

In the case of a conic the chord joining point pairs in involution (11) passes through a fixed point. The tangents at the fixed points of the involution being self-perpendicular pass through the circular points. Hence the ortho-chord always passes through a focus and the corresponding directrix is the orthoptic locus.

Consider next a curve of class three for which the line at infinity is a bi-tangent at two points say A and B. Take any fixed tangent t to the curve. If T be any point on t we may draw from T two tangents to the curve other than t and these meet the line at infinity in say π and π'. It is easy to see that π and π' are in one-one correspondence, each being the conjugate of the other. We have thus established an involution on the line at infinity the double points of which are the points at infinity on the two tangents to the curve whose chord of contact is t. If these be the circular points, that is, if t be the *directrix*—chord of contact of tangents from I and J to the curve—π and π' will separate I and J harmonically and hence tangents from T to the curve will be at right angles. Again, in the involution $\pi\ \pi'$, the points A and B form a corresponding pair since the tangents to the curve at these points—both coincident with the line at infinity—fulfil the condition of meeting on t. Hence A and B separate harmonically I and J, that is, the line at infinity must be a bi-tangent, the points of contact being at infinity in perpendicular directions. This condition ensures that I and J shall be the double points of the involution, that is, that t shall be the orthoptic locus.

"On Some Infinite Series and Products."

By M. Bhimasena Rao and M. Venkatarama Ayyar.

Part I.*

This paper deals with the evaluation of certain infinite series and products. A few of them have already been evaluated by Dr. Glaisher and others; but, their mode of evaluation is complex, involving a knowledge of elliptic functions and modular equations. The method adopted here is fairly elementary, consisting of simple transformations of two well-known infinite integrals and is believed to be new.†

§ 1. It is well-known that‡

$$\int_0^\infty \frac{\sin(bx)\,dx}{e^{2\pi x}-1} = \frac{1}{2}\left(\frac{1}{e^b-1}-\frac{1}{b}+\frac{1}{2}\right), \text{ if } b = p+iq,\ |q| < 2\pi.$$

$$\therefore \int_0^\infty \frac{4\sin(xt)\cos(2ant)}{e^{2\pi t}-1}\,dt = \int_0^\infty \frac{2\sin(2an+x)t}{e^{2\pi t}-1}\,dt - \int_0^\infty \frac{2\sin(2an-x)t}{e^{2\pi t}-1}\,dt$$

$$= \left[\frac{1}{e^{2an+x}-1}-\frac{1}{2an+x}+\frac{1}{2}\right]-\left[\frac{1}{e^{2an-x}-1}-\frac{1}{2an-x}+\frac{1}{2}\right]$$

where the imaginary part of $(2an \pm x) < 2\pi$ in absolute value.

$$\therefore \int_0^\infty \frac{4\sin(xt)\sum_1^n \cos(2ant)\,dt}{e^{2\pi t}-1}$$

$$= \sum_1^n \Big[\{e^{-(2an+x)} + e^{-2(2an+x)} + e^{-3(2an+x)} + \ldots\ldots\} - \{e^{-(2an-x)} + e^{-2(2an-x)} + e^{-3(2an-x)} + \ldots\ldots\} + \frac{2x}{4a^2n^2-x^2}\Big],$$

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* Read before the Eleventh Session of the Indian Science Congress held at Bangalore in January 1924.

† In this connection, it may be mentioned that the authors have examined the first 45 Volumes of the *Messenger of Mathematics*, the *Fundamenta Nova* of Jacobi, the Treatises on *Elliptic Functions* by Cayley, Greenhill and Hancock, and a few stray issues of the *Quarterly Journal of Mathematics*.

‡ *Vide* Bromwich: *Introduction to Infinite Series*, p. 454, Ex. 1.

which, on expanding each term in the curvilinear brackets, writing in rows one underneath the other and adding column by column, becomes

$$= \left\{ e^{-(2a+x)} \cdot \frac{1-e^{-2na}}{1-e^{-2a}} + e^{-2(2a+x)} \cdot \frac{1-e^{-2n.2a}}{1-e^{-2\,2a}} + \ldots\ldots \right\}$$

$$- \left\{ e^{-(2a-x)} \cdot \frac{1-e^{-2na}}{1-e^{-2a}} + e^{-2(2a-x)} \cdot \frac{1-e^{-2n.2a}}{1-e^{-2\,2a}} + \ldots\ldots \right\}$$

$$+ \sum_1^n \frac{2x}{4a^2n^2 - x^2}.$$

Now, $$\cot x = \frac{1}{x} + \sum_1^\infty \frac{2x}{x^2 - n^2\pi^2}.$$

$$\therefore \quad \sum_1^\infty \frac{2x}{4a^2n^2 - x^2} = \frac{1}{x} - \frac{\pi}{2a} \cot \frac{\pi x}{2a}.$$

In the result above, use this after letting n tend to infinity; then

$$\lim_{n \to \infty} \int_0^\infty \frac{4 \sin xt \sum_1^n \cos 2ant}{e^{2\pi t} - 1}\, dt$$

$$= \left\{ \frac{e^{-x}}{e^{2a} - 1} + \frac{e^{-2x}}{e^{4a} - 1} + \frac{e^{-3x}}{e^{6a} - 1} + \ldots\ldots \right\}$$

$$- \left\{ \frac{e^{x}}{e^{2a} - 1} + \frac{e^{2x}}{e^{4a} - 1} + \frac{e^{3x}}{e^{6a} - 1} + \ldots\ldots \right\}$$

$$+ \frac{1}{x} - \frac{\pi}{2a} \cot \left(\frac{\pi x}{2a} \right).$$

$$= -2 \left\{ \frac{\sinh x}{e^{2a} - 1} + \frac{\sinh 2x}{e^{4a} - 1} + \frac{\sinh 3x}{e^{6a} - 1} + \ldots\ldots \right\}$$

$$+ \frac{1}{x} - \frac{\pi}{2a} \cot \left(\frac{\pi x}{2a} \right). \quad \ldots \qquad \ldots \quad (a)$$

Now, $$\sum_1^n 2 \cos 2ant = \frac{\sin (2n+1)at}{\sin at} - 1.$$

$$\therefore \quad \int_0^\infty \frac{4 \sin xt \sum_1^n \cos 2ant}{e^{2\pi t} - 1}\, dt = \int_0^\infty \frac{\sin (2n+1)at}{\sin at} \cdot \frac{2 \sin xt}{e^{2\pi t} - 1}\, dt$$

$$- \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} \right).$$

Here let n tend to infinity. In the integral on the right-hand side put $at = z$ and use the well-known result :*

$$\lim_{n \to \infty} \int_0^\infty \frac{\sin(2n+1)z}{\sin z} f(z)\, dz = \frac{\pi}{2}\left[f(0) + 2f(\pi) + 2f(2\pi) + \ldots\ldots\right].$$

We have

$$\lim_{n \to \infty} \int_0^\infty \frac{4 \sin xt \sum_1^n \cos 2ant}{e^{2\pi t} - 1}\, dt$$

$$= \frac{\pi}{2a}\left\{\frac{x}{\pi} + \frac{4 \sin\left(\frac{x\pi}{a}\right)}{e^{2\pi^2/a} - 1} + \frac{4 \sin\left(\frac{2x\pi}{a}\right)}{e^{4\pi^2/a} - 1} + \ldots\ldots\right\}$$

$$- \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2}\right). \qquad \ldots \quad (b)$$

§ 2. From (a) and (b), we get

$$\left\{\frac{\sinh x}{e^{2a} - 1} + \frac{\sinh 2x}{e^{4a} - 1} + \frac{\sinh 3x}{e^{6a} - 1} + \ldots\ldots\right\}$$

$$+ \frac{\pi}{a}\left\{\frac{\sin\left(\frac{\pi x}{a}\right)}{e^{2\pi^2/a} - 1} + \frac{\sin\left(\frac{2\pi x}{a}\right)}{e^{4\pi^2/a} - 1} + \ldots\ldots\right\}$$

$$= \frac{1}{2}\left\{\coth\frac{1}{2}\left(\frac{x}{2}\right) - \frac{\pi}{2a}\cot\left(\frac{\pi x}{2a}\right) - \frac{x}{2a}\right\}. \qquad (c)$$

Here, equate the co-efficients of x^{2n+1} on both sides. Then,

(i) when $n \geqslant 1$,

$$\left\{\frac{1^{2n+1}}{e^{2a} - 1} + \frac{2^{2n+1}}{e^{4a} - 1} + \frac{3^{2n+1}}{e^{6a} - 1} + \ldots\ldots\right\}$$

$$+ (-1)^n \left(\frac{\pi}{a}\right)^{2n+2} \left\{\frac{1^{2n+1}}{e^{\frac{2\pi^2}{a}} - 1} + \frac{2^{2n+1}}{e^{\frac{4\pi^2}{a}} - 1} + \frac{3^{2n+1}}{e^{\frac{6\pi^2}{a}} - 1} + \ldots\ldots\right\}$$

$$= \frac{B_{n+1}}{4(n+1)} \cdot \left\{\left(\frac{\pi}{a}\right)^{2n+2} + (-1)^n\right\}. \qquad \ldots \quad (1)$$

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* *Vide* Bromwich : *Introduction to Infinite Series* : p. 447, Ex. 4.

and, (ii) when $n = 0$,

$$\left\{\frac{1}{e^{2a}-1}+\frac{2}{e^{4a}-1}+\frac{3}{e^{6a}-1}+\ldots\ldots\right\}$$

$$+\left(\frac{\pi}{a}\right)^2\left\{\frac{1}{e^{\frac{2\pi^2}{a}}-1}+\frac{2}{e^{\frac{4\pi^2}{a}}-1}+\frac{3}{e^{\frac{6\pi^2}{a}}-1}+\ldots\ldots\right\}$$

$$=\frac{1}{24}\left(\frac{\pi^2}{a^2}+1\right)-\frac{1}{4a}. \qquad \ldots \qquad \ldots (1.1)$$

In (1), let n be an even integer equal to $2m$ (say) and $a = \pi$.* Then,

$$\frac{1^{4m+1}}{e^{2\pi}-1}+\frac{2^{4m+1}}{e^{4\pi}-1}+\frac{3^{4m+1}}{e^{6\pi}-1}+\ldots\ldots=\frac{B_{2m+1}}{4(2m+1)}. \quad \ldots \quad (2)$$

Similarly, in (1.1), put $a = \pi$, we have

$$\frac{1}{e^{2\pi}-1}+\frac{2}{e^{4\pi}-1}+\frac{3}{e^{6\pi}-1}+\ldots\ldots=\frac{1}{24}-\frac{1}{8\pi}. \qquad \ldots (3)$$

a problem proposed by the late S. Ramanujan, F.R.S.†

§ 3. Now, in (1), change a into $\frac{a}{2}$. Then,

$$\left\{\frac{1^{2n+1}}{e^{a}-1}+\frac{2^{2n+1}}{e^{2a}-1}+\frac{3^{2n+1}}{e^{3a}-1}+\ldots\ldots\right\}$$

$$+(-1)^n\left(\frac{\pi}{a}\right)^{2n+2}2^{2n+2}\left\{\frac{1^{2n+1}}{e^{\frac{4\pi^2}{a}}-1}+\frac{2^{2n+1}}{e^{\frac{8\pi^2}{a}}-1}+\frac{3^{2n+1}}{e^{\frac{12\pi^2}{a}}-1}+\ldots\ldots\right\}$$

$$=\frac{B_{n+1}}{4(n+1)}\cdot\left\{\left(\frac{\pi}{a}\right)^{2n+2}\cdot 2^{2n+2}+(-1)^n\right\}. \quad \ldots \quad (d)$$

Subtract twice (1) from (*d*). Then

$$\left\{\frac{1^{2n+1}}{e^{a}+1}+\frac{2^{2n+1}}{e^{2a}+1}+\frac{3^{2n+1}}{e^{3a}+1}+\ldots\ldots\right\}$$

$$+(-1)^{n+1}2\left(\frac{\pi}{a}\right)^{2n+2}\left\{\frac{1^{2n+1}}{e^{\frac{2\pi^2}{a}}-1}+\frac{3^{2n+1}}{e^{\frac{6\pi^2}{a}}-1}+\frac{5^{2n+1}}{e^{\frac{10\pi^2}{a}}-1}+\ldots\ldots\right\}$$

$$=\frac{B_{n+1}}{4(n+1)}\left\{\left(\frac{\pi}{a}\right)^{2n+2}\cdot 2\cdot(2^{2n+1}-1)+(-1)^{n+1}\right\} \quad \ldots \quad (e)$$

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* *Vide* one of the results in Art. 94, p. 81, Vol. 18 of the *Messenger of Mathematics* (1889).

† *Vide* J.I.M.S., Vol. IV, Question No. 387.

Again, subtract 2^{2n+1} times (1) from (*d*); we obtain

$$\left\{\frac{1^{2n+1}}{e^{a}-1}+\frac{3^{2n+1}}{e^{3a}-1}+\frac{5^{2n+1}}{e^{5a}-1}+\ldots\ldots\right\}$$

$$+(-1)^{n+1}\left(\frac{\pi}{a}\right)^{2n+2}\left\{\frac{2^{2n+1}}{e^{\frac{2\pi^2}{a}}+1}+\frac{4^{2n+1}}{e^{\frac{4\pi^2}{a}}+1}+\frac{6^{2n+1}}{e^{\frac{6\pi^2}{a}}+1}+\ldots\ldots\right\}$$

$$=\frac{B_{n+1}}{4(n+1)}\left\{\left(\frac{\pi}{a}\right)^{2n+2}.2^{2n+1}+(-1)^{n+1}.(2^{2n+1}-1)\right\} \quad \ldots \quad (f)$$

In (*f*), change a into $\frac{\pi^2}{a}$. We get

$$\left\{\frac{2^{2n+1}}{e^{2a}+1}+\frac{4^{2n+1}}{e^{4a}+1}+\frac{6^{2n+1}}{e^{6a}+1}+\ldots\ldots\right\}$$

$$=(-1)^{n}\left(\frac{\pi}{a}\right)^{2n+2}\left\{\frac{1^{2n+1}}{e^{\frac{\pi^2}{a}}-1}+\frac{3^{2n+1}}{e^{\frac{3\pi^2}{a}}-1}+\frac{5^{2n+1}}{e^{\frac{5\pi^2}{a}}-1}+\ldots\ldots\right\}$$

$$+(-1)^{n+1}\frac{B_{n+1}}{4(n+1)}\left\{2^{2n+1}+(-1)^{n+1}\left(\frac{\pi}{a}\right)^{2n+2}(2^{2n+1}-1)\right\}. \quad (g)$$

From (*g*), (*f*) and (*e*), we finally arrive at

$$\left\{\frac{1^{2n+1}}{e^{a}+1}+\frac{3^{2n+1}}{e^{3a}+1}+\frac{5^{2n+1}}{e^{5a}+1}+\ldots\ldots\right\}$$

$$+(-1)^{n}\left(\frac{\pi}{a}\right)^{2n+2}\left\{\frac{1^{2n+1}}{e^{\frac{\pi^2}{a}}+1}+\frac{3^{2n+1}}{e^{\frac{3\pi^2}{a}}+1}+\frac{5^{2n+1}}{e^{\frac{5\pi^2}{a}}+1}+\ldots\right\}$$

$$=\frac{B_{n+1}}{4(n+1)}(2^{2n+1}-1)\left\{\left(\frac{\pi}{a}\right)^{2n+2}+(-1)^{n}\right\}, \ldots (4)$$

a result true for $n \geq 1$ and also for $n = 0$, as starting with (1.1), we will arrive only at the same result.

Here, put $n = 2m$ and $a = \pi$. Then,

$$\frac{1^{4m+1}}{e^{\pi}+1}+\frac{3^{4m+1}}{e^{3\pi}+1}+\frac{5^{4m+1}}{e^{5\pi}+1}+\ldots\ldots$$

$$=\frac{B_{2m+1}}{4(2m+1)}.(2^{4m+1}-1). \quad \ldots \quad (5)$$

§ 4. Now, take result (1), change a into πa therein and differentiate with respect to a. We get

$$-\frac{\pi}{2}\{\operatorname{cosech}^2(\pi a)+2^{2n+2}\operatorname{cosech}^2(2\pi a)+\ldots\ldots\}$$

$$+(-1)^n\frac{\pi}{2}\frac{1}{a^{2n+4}}\left\{\operatorname{cosech}^2\left(\frac{\pi}{a}\right)+2^{2n+2}\operatorname{cosech}^2\left(\frac{2\pi}{a}\right)+\ldots\ldots\right\}$$

$$+(-1)^{n+1}\cdot\frac{2n+2}{a^{2n+3}}\cdot\left\{\frac{1^{2n+1}}{e^{\frac{2\pi}{a}}-1}+\frac{2^{2n+1}}{e^{\frac{4\pi}{a}}-1}+\frac{3^{2n+1}}{e^{\frac{6\pi}{a}}-1}+\ldots\ldots\right\}$$

$$=\frac{B_{n+1}}{4(n+1)}\cdot\frac{-(2n+2)}{a^{2n+3}}.$$

Here put $n=2m-1$, m being any positive integer, and $a=1$. Then

$$\pi\{\operatorname{cosech}^2(\pi)+2^{4m}\operatorname{cosech}^2(2\pi)+3^{4m}\operatorname{cosech}^2(3\pi)+\ldots\ldots\}$$

$$=4m\left\{\frac{B^{2m}}{8m}+\frac{1^{4m-1}}{e^{2\pi}-1}+\frac{2^{4m-1}}{e^{4\pi}-1}+\frac{3^{4m-1}}{e^{6\pi}-1}+\ldots\ldots\right\},$$

a result due to Ramanujan* ... ... (6)

In the same way, take result (4), change a into πa and differentiate with respect to a. We obtain

$$-\frac{\pi}{4}\left\{\operatorname{sech}^2\left(\frac{\pi a}{2}\right)+3^{2n+2}\operatorname{sech}^2\left(\frac{3\pi a}{2}\right)+\ldots\ldots\right\}$$

$$+(-1)^n\cdot\frac{\pi}{4}\cdot\frac{1}{a^{2n+4}}\left\{\operatorname{sech}^2\left(\frac{\pi}{2a}\right)+3^{2n+2}\operatorname{sech}^2\left(\frac{3\pi}{2a}\right)+\ldots\ldots\right\}$$

$$+(-1)^{n+1}\cdot\frac{2n+2}{a^{2n+3}}\cdot\left\{\frac{1}{e^{\frac{\pi}{a}}+}+\frac{3^{2n+1}}{e^{\frac{3\pi}{a}}+1}+\frac{5^{2n+1}}{e^{\frac{5\pi}{a}}+1}+\ldots\ldots\right\}$$

$$=\frac{B_{n+1}}{4(n+1)}(2^{2n+1}-1)\cdot\frac{-(2n+2)}{a^{2n+3}}.$$

Here, let $n=2m-1$, m being any positive integer and put $a=1$. Then,

$$\frac{\pi}{2}\left\{\operatorname{sech}^2\left(\frac{\pi}{2}\right)+3^{4m}\operatorname{sech}^2\left(\frac{3\pi}{2}\right)+5^{4m}\operatorname{sech}^2\left(\frac{5\pi}{2}\right)+\ldots\ldots\right\}$$

$$=4m\left\{\frac{B_{2m}(2^{4m-1}-1)}{8m}+\frac{1^{4m-1}}{e^{\pi}+1}+\frac{3^{4m-1}}{e^{3\pi}+1}+\ldots\ldots\right\}$$

a result similar to (6). ... ... (7)

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* *Vide* J. I. M. S., Vol. XIV, No. 3, page 89, result 8.

§ 5. In the result (4), put $n = 0$. Then,

$$\left\{\frac{1}{e^a+1}+\frac{3}{e^{3a}+1}+\frac{5}{e^{5a}+1}+\ldots\ldots\right\}$$
$$+\left(\frac{\pi}{a}\right)\cdot\left\{\frac{1}{e^{\frac{\pi^2}{a}}+1}+\frac{3}{e^{\frac{3\pi^2}{a}}+1}+\frac{5}{e^{\frac{5\pi^2}{a}}+1}+\ldots\ldots\right\}$$
$$=\frac{1}{24}\left(\frac{\pi^2}{a^2}+1\right).\quad\ldots\qquad\ldots\quad(h)$$

Integrate both sides with respect to a. Then,

$$[-\log(1+e^{-a})-\log(1+e^{-3a})-\log(1+e^{-5a})-\ldots\ldots]$$
$$+\left[\log\left(1+e^{-\frac{\pi^2}{a}}\right)+\log\left(1+e^{-\frac{3\pi^2}{a}}\right)+\ldots\ldots\right]$$
$$=-\frac{1}{24}\left(\frac{\pi^2}{a}-a\right)+\text{a constant.}$$

On removing the logarithms, changing a into πa and inverting the fraction,

$$\frac{(1+e^{-\pi a})(1+e^{-3\pi a})(1+e^{-5\pi a})\ldots}{\left(1+e^{-\frac{\pi}{a}}\right)\left(1+e^{-\frac{3\pi}{a}}\right)\left(1+e^{-\frac{5\pi}{a}}\right)\ldots}$$
$$=Ae^{\frac{\pi}{24}\left(\frac{1}{a}-a\right)}=e^{-\frac{\pi}{24}\left(a-\frac{1}{a}\right)},$$

A being equal to 1 as is easily seen by putting $a = 1$.

Similarly, by taking result (1.1), integrating both sides with respect to a and proceeding as before, the corresponding result is

$$\frac{(1-e^{-2\pi a})(1-e^{-4\pi a})(1-e)^{-6\pi a})\ldots}{\left(1-e^{-\frac{2\pi}{a}}\right)\left(1-e^{-\frac{4\pi}{a}}\right)\left(1-e^{-\frac{6\pi}{a}}\right)\ldots}=a^{-\frac{1}{2}}.e^{\frac{\pi}{12}\left(a-\frac{1}{a}\right)}.\ \ldots\quad(9)$$

In this, put $a=\frac{1}{\sqrt{2}}$. Then,

$$(1-e^{-\pi\sqrt{2}})(1-e^{-3\pi\sqrt{2}})(1-e^{-5\sqrt{2}})\ldots\ldots=2^{\frac{1}{4}}e^{-\frac{\pi\sqrt{2}}{24}}.\qquad\ldots\quad(10)$$

§ 6. In (9), put $a = r(\cos\theta + i\sin\theta)$ and take the logarithms of both sides. Then,

$$\log\left[\frac{(A + iB) \times \ldots\ldots}{(A' - iB') \times \ldots\ldots}\right]$$

$$= \frac{\pi}{12}\left\{r(\cos\theta + i\sin\theta) - \frac{1}{r}(\cos\theta - i\sin\theta)\right\} - \tfrac{1}{2}\log(re^{i\theta})\ldots \quad \text{(i)}$$

where

$$A \equiv 1 + e^{-2\pi r\cos\theta}.\cos(2\pi r\sin\theta);\ B \equiv e^{-2\pi r\cos\theta}.\sin(2\pi r\sin\theta);$$

$$A' \equiv 1 - e^{-\frac{2\pi}{r}\cos\theta}.\cos\left(\frac{2\pi}{r}\sin\theta\right);\ B' \equiv e^{-\frac{2\pi}{r}\cos\theta}.\sin\left(\frac{2\pi}{r}\sin\theta\right).$$

Separate this into real and imaginary parts and equate the real parts.

$$\tfrac{1}{2}[\log\{1 - 2e^{-2\pi r\cos\theta}\cos(2\pi r\sin\theta) + e^{-4\pi r\cos\theta}\}$$
$$+ \log\{1 - 2e^{-4\pi r\cos\theta}\cos(4\pi r\sin\theta) + e^{-8\pi r\cos\theta}\}$$
$$+ \ldots\ldots\ldots]$$
$$-\tfrac{1}{2}\left[\log\left\{1 - 2e^{-\frac{2\pi}{r}\cos\theta}\cos\left(\frac{2\pi}{r}\sin\theta\right) + e^{-\frac{4\pi}{r}\cos\theta}\right\}\right.$$
$$+ \log\left\{1 - 2e^{-\frac{4\pi}{r}\cos\theta}\cos\left(\frac{4\pi}{r}\sin\theta\right) + e^{-\frac{8\pi}{r}\cos\theta}\right\}$$
$$\left. + \ldots \quad \ldots\right]$$
$$= \frac{\pi\cos\theta}{12}\left(r - \frac{1}{r}\right) - \frac{1}{2}\log r. \quad \ldots \qquad \ldots \quad (j)$$

Here put $r = \sqrt{2}$, cancel the terms of the first part of the left-hand side with the even terms of the second part. We get

$$-\tfrac{1}{2}[\log\{1 - 2e^{-\pi\sqrt{2}\cos\theta}\cos(\pi\sqrt{2}\sin\theta) + e^{-2\pi\sqrt{2}\cos\theta}\}$$
$$+ \log\{1 - 2e^{-3\pi\sqrt{2}\cos\theta}\cos(3\pi\sqrt{2}\sin\theta) + e^{-6\pi\sqrt{2}\cos\theta}\}$$
$$+ \ldots\ldots\ldots] = \frac{\pi\cos\theta}{12\sqrt{2}} - \frac{1}{4}\log 2. \qquad \ldots \quad (k)$$

Here put $\sin\theta = \frac{1}{\sqrt{2}} = \cos\theta$. Then, we have

$$-\tfrac{1}{2}[\log(1 + 2e^{-\pi} + e^{-2\pi}) + \log(1 + 2e^{-3\pi} + e^{-6\pi}) + \ldots\ldots]$$
$$= \frac{\pi}{24} - \frac{1}{4}\log 2.$$

Or, $\quad (1 + e^{-\pi})(1 + e^{-3\pi})(1 + e^{-5\pi})\ldots\ldots = 2^{\frac{1}{4}}e^{-\frac{\pi}{24}}$ * $\quad \ldots \quad (11)$

* Compare result 28, p. 178, Vol. 5 of the *Messenger of Mathematics* (1876).

In (k), put $\sin\theta = \frac{1}{2\sqrt{2}}$ and $\cos\theta = \frac{\sqrt{7}}{2\sqrt{2}}$. Then,

$$(1+e^{-\pi\sqrt{7}})(1+e^{-3\pi\sqrt{7}})(1+e^{-5\pi\sqrt{7}})\ldots\ldots = 2^{\frac{1}{8}}e^{-\frac{\pi\sqrt{7}}{24}}. \qquad (12)$$

Now, in (8), put $a = \sqrt{7}$ and substitute from (12); then

$$(1+e^{-\frac{\pi}{\sqrt{7}}})(1+e^{-\frac{3\pi}{\sqrt{7}}})(1+e^{-\frac{5\pi}{\sqrt{7}}})\ldots\ldots = 2^{\frac{1}{8}}e^{-\frac{\pi}{24\sqrt{7}}}. \quad \ldots \quad (13)$$

Again, in (9) change a into $\frac{a}{2}$; divide (9) by the result so got, we get

$$(1+e^{-\pi a})(1+e^{-2\pi a})(1+e^{-3\pi a})\ldots\ldots$$

$$\times(1+e^{-\frac{2\pi}{a}})(1+e^{-\frac{4\pi}{a}})(1+e^{-\frac{6\pi}{a}})\ldots\ldots = 2^{-\frac{1}{8}}e^{\frac{\pi}{24}\left(a+\frac{2}{a}\right)}. \quad (l)$$

In this, put $a = \sqrt{2}$ and take the square root,

$$(1+e^{-\pi\sqrt{2}})(1+e^{-2\pi\sqrt{2}})(1+e^{-3\pi\sqrt{2}})\ldots\ldots = 2^{-\frac{1}{4}}e^{\frac{\pi\sqrt{2}}{24}}. \qquad (14)$$

Again in (l), put $a = 1$. Then,

$$(1+e^{-\pi})(1+e^{-2\pi})^2(1+e^{-3\pi})(1+e^{-4\pi})^2\ldots\ldots = 2^{-\frac{1}{2}}e^{\frac{\pi}{8}}.$$

But, in result (10), we are given the value of $(1+e^{-\pi})(1+e^{-3\pi})\ldots\ldots$ Hence, using it here,

$$(1+e^{-2\pi})(1+e^{-4\pi})(1+e^{-6\pi})\ldots\ldots = 2^{-\frac{3}{8}}e^{\frac{\pi}{12}}. \quad \ldots \quad (15)$$

It may be noticed, in passing, that, on combining results (11) and (15),

$$(1+e^{-\pi})(1+e^{-2\pi})(1+e^{-3\pi})(1+e^{-4\pi})\ldots\ldots = 2^{-\frac{1}{8}}e^{\frac{\pi}{24}}.$$

Now, in result (l), put $a = r(\cos\theta + i\sin\theta)$, take the logarithms of both sides, separate into real and imaginary parts and equate the real parts. We get

$$\begin{aligned}
&\tfrac{1}{2}\big[\log\{1+2e^{-\pi r\cos\theta}.\cos(\pi r\sin\theta)+e^{-2\pi r\cos\theta}\}\\
&\quad+\log\{1+2e^{-2\pi r\cos\theta}\cos(2\pi r\sin\theta)+e^{-4\pi r\cos\theta}\}\\
&\quad\quad+\ldots\ldots\ldots\ldots\ldots\\
&+\log\{1+2e^{-\frac{2\pi}{r}\cos\theta}\cos\left(\frac{2\pi}{r}\sin\theta\right)+e^{-\frac{4\pi}{r}\cos\theta}\}\\
&\quad+\log\{1+2e^{-\frac{4\pi}{r}\cos\theta}\cos\left(\frac{4\pi}{r}\sin\theta\right)+e^{-\frac{8\pi}{r}\cos\theta}\}\\
&\quad\quad+\ldots\ldots\ldots\ldots\ldots\big]\\
&\quad=-\frac{1}{2}\log 2+\frac{\pi\cos\theta}{24}\left(r+\frac{2}{r}\right). \qquad \ldots \quad (m)
\end{aligned}$$

In this, put $r = 1$, $\sin\theta = \frac{1}{2}$. Then,

$$\tfrac{1}{2}\log(1+e^{-\pi\sqrt{3}})(1-e^{-\pi\sqrt{3}})^2(1+e^{-3\pi\sqrt{3}})(1+e^{-2\pi\sqrt{3}})^2\ldots.$$
$$+\tfrac{1}{2}\log(1-e^{-\pi\sqrt{3}})^2(1+e^{2-\pi\sqrt{3}})^2(1-e^{-3\pi\sqrt{3}})^2\ldots\ldots\ldots\ldots$$
$$= -\tfrac{1}{2}\log 2 + \frac{\pi\sqrt{3}}{16}.$$

i.e., $$(1+e^{-\pi\sqrt{3}})(1-e^{-\pi\sqrt{3}})^4(1+e^{-3\pi\sqrt{3}})(1+e^{-2\pi\sqrt{3}})^4\ldots\ldots$$
$$= 2^{-1}e^{\frac{\pi\sqrt{3}}{8}}.$$

On putting $e^{-\pi\sqrt{3}} = q$, we have

$$(1+q)(1-q)^4(1+q^3)(1+q^2)^4(1+q^5)(1-q^3)^4\ldots\ldots = 2^{-1}q^{-\frac{1}{8}}.$$

But it is well-known that

$$\prod_1^\infty(1-q^{2n-1})\prod_1^\infty(1+q^{2n})\prod_1^\infty(1+q^{2n-1}) = 1.$$

Using this, we get

$$\frac{1}{(1+q)^3(1+q^3)^3(1+q^5)^3\ldots\ldots} = 2^{-1}q^{-\frac{1}{8}}.$$

$$\therefore\quad (1+e^{-\pi\sqrt{3}})(1+e^{-3\pi\sqrt{3}})(1+e^{-5\pi\sqrt{3}})\ldots\ldots = 2^{\frac{1}{3}}e^{-\frac{\pi\sqrt{3}}{24}}.\ *\quad (16)$$

Putting $a = \frac{1}{\sqrt{3}}$ in (8), we get

$$(1+e^{-\frac{\pi}{\sqrt{3}}})(1+e^{-\frac{3\pi}{\sqrt{3}}})(1+e^{-\frac{5\pi}{\sqrt{3}}})\ldots\ldots = 2^{\frac{1}{3}}e^{-\frac{\pi}{24\sqrt{3}}}.\ \dagger\quad \ldots\quad (17)$$

Now, following the notation in Bromwich's *Infinite Series*, pp. 105—107, we may use the equations

$$Q_1Q_2Q_3 = 1 \text{ and } Q_2^8 = Q_3^8 + 16qQ_1^8$$

and derive the values of infinite products related to those got above. A small table of such values is given as an Appendix to this part of the paper.

§ 7. Take result (1) and in it, put $a = \pi r(\cos\theta + i\sin\theta)$. Then, on reducing, splitting into real and imaginary parts, and equating the real parts,

$$\Bigg[\frac{e^{2\pi r\cos\theta}.\cos(2\pi r\sin\theta)-1}{e^{4\pi r\cos\theta}-2e^{2\pi r\cos\theta}\cos(2\pi r\sin\theta)+1} + 2^{2n+1}\frac{e^{4\pi r\cos\theta}\cos(4\pi r\sin\theta)-1}{e^{8\pi r\cos\theta}-2e^{4\pi r\cos\theta}\cos(4\pi r\sin\theta)+1}+\ldots\ldots\Bigg]$$

* *Vide*: Result No. 26, p. 178, Vol. 5, *Messenger of Mathematics* (1876).

† " " 27, " " "

$$+(-1)^n \cdot \frac{\cos(2n+2)\theta}{r^{2n+2}}\left\{\frac{e^{\frac{2\pi}{r}\cos\theta}\cos\left(\frac{2\pi}{r}\sin\theta\right)-1}{e^{\frac{4\pi}{r}\cos\theta}-2e^{\frac{2\pi}{r}\cos\theta}\cos\left(\frac{2\pi}{r}\sin\theta\right)+1}\right.$$

$$\left.+\,2^{2n+1}\frac{e^{\frac{4\pi}{r}\cos\theta}\cos\left(\frac{4\pi}{r}\sin\theta\right)-1}{e^{\frac{8\pi}{r}\cos\theta}-2e^{\frac{4\pi}{r}\cos\theta}\cos\left(\frac{4\pi}{r}\sin\theta\right)+1}+\ldots\ldots\right\}$$

$$+(-1)^n \cdot \frac{\sin(2n+2)\theta}{r^{2n+2}}\left\{\frac{e^{\frac{2\pi}{r}\cos\theta}\sin\left(\frac{2\pi}{r}\sin\theta\right)}{e^{\frac{4\pi}{r}\cos\theta}-2e^{\frac{2\pi}{r}\cos\theta}\cos\left(\frac{2\pi}{r}\sin\theta\right)+1}\right.$$

$$\left.+\,2^{2n+1}\frac{e^{\frac{4\pi}{r}\cos\theta}\sin\left(\frac{4\pi}{r}\sin\theta\right)}{e^{\frac{8\pi}{r}\cos\theta}-2e^{\frac{4\pi}{r}\cos\theta}\cos\left(\frac{4\pi}{r}\sin\theta\right)+1}+\ldots\ldots\right\}$$

$$=\frac{B_{n+1}}{4(n+1)}\cdot\left\{\frac{\cos(2n+2)\theta}{r^{2n+2}}+(-1)^n\right\}. \quad \ldots \quad (n)$$

Here, put $r=1$, $\sin\theta=\frac{1}{2}$. We get

$$\left\{\frac{-e^{\pi\sqrt{3}}-1}{(e^{\pi\sqrt{3}}+1)^2}+2^{2n+1}\cdot\frac{e^{2\pi\sqrt{3}}-1}{(e^{2\pi\sqrt{3}}-1)^2}+3^{2n+1}\cdot\frac{-e^{3\pi\sqrt{3}}-1}{(e^{3\pi\sqrt{3}}+1)^2}+\ldots\ldots\right\}$$

$$\times\left\{1+(-1)^n\cos(n+1)\frac{\pi}{3}\right\}$$

$$=\frac{B_{n+1}}{4(n+1)}\left\{\cos(n+1)\frac{\pi}{3}+(-1)^n\right\}. \ldots(o)$$

Here, the common factor $\left[\cos(n+1)\frac{\pi}{3}+(-1)^n\right]$ can be removed from both sides, whenever n is not of the form $3m-1$.

Hence, when $n=0$ or 1 (modulus 3),

$$\frac{1^{2n+1}}{e^{\pi\sqrt{3}}+1}-\frac{2^{2n+1}}{e^{2\pi\sqrt{3}}-1}+\frac{3^{2n+1}}{e^{3\pi\sqrt{3}}+1}-\frac{4^{2n+1}}{e^{4\pi\sqrt{3}}-1}+\ldots\ldots$$

$$=(-1)^{n+1}\frac{B_{n+1}}{4(n+1)}\ldots \quad (18)$$

Proceeding in a similar manner with result (1.1) and making the same substitutions for r and θ, we obtain

$$\frac{1}{e^{\pi\sqrt{3}}+1}-\frac{2}{e^{2\pi\sqrt{3}}-1}+\frac{3}{e^{3\pi\sqrt{3}}+1}-\frac{4}{e^{4\pi\sqrt{3}}-1}+\ldots\ldots\ldots$$

$$=\frac{\sqrt{3}}{12\pi}-\frac{1}{24}. \qquad \ldots (19)$$

Again, in (o), put $r=\sqrt{2}$., $\sin\theta=\frac{1}{\sqrt{2}}$ and $\cos\theta=\frac{1}{\sqrt{2}}$. Then,

$$\left[\frac{(e^{2\pi}-1)}{(e^{2\pi}-1)^2}+2^{2n+1}.\frac{(e^{4\pi}-1)}{(e^{4\pi}-1)^2}+3^{2n+1}.\frac{(e^{6\pi}-1)}{(e^{6\pi}-1)^2}+\ldots\ldots\ldots\right]$$

$$+\frac{1}{2^{n+1}}.(-1)^n\cos(n+1)\frac{\pi}{2}.\left[\frac{(-e^{\pi}-1)}{(e^{\pi}+1)^2}+\frac{(e^{2\pi}-1)}{(e^{2\pi}-1)^2}.2^{2n+1}\right.$$

$$\left.+\frac{(-e^{3\pi}-1)}{(e^{3\pi}+1)^2}.3^{2n+1}+\ldots\ldots\ldots\right]$$

$$=\frac{B_{n+1}}{4(n+1)^n}\left\{\frac{\cos(n+1)\frac{\pi}{2}}{2^{n+1}}+(-1)^n\right\}. \qquad \ldots (p)$$

Here, let $n=1$. Then,

$$\left(\frac{1^3}{e^{3\pi}-1}+\frac{2^3}{e^{4\pi}-1}+\frac{3^3}{e^{6\pi}-1}+\ldots\ldots\right)$$

$$+\tfrac{1}{4}\left(\frac{-1^3}{e^{\pi}+1}+\frac{2^3}{e^{2\pi}-1}+\frac{-3^3}{e^{3\pi}+1}+\ldots\ldots\ldots\right)$$

$$=\frac{B_2}{4.2}.(-\tfrac{1}{4}+1).$$

$$\therefore\ 3\left(\frac{1^3}{e^{2\pi}-1}+\frac{2^3}{e^{4\pi}-1}+\frac{3^3}{2^{6\pi}-1}+\ldots\ldots\ldots\right)$$

$$-\tfrac{1}{4}\left(\frac{1}{e^{\pi}+1}+\frac{3^3}{e^{3\pi}+1}+\frac{5^3}{e^{5\pi}+1}+\ldots\ldots\right)$$

$$=-\frac{1}{48\times 4}.$$

or,
$$\frac{1}{e^{\pi}+1}+\frac{3^3}{e^{3\pi}+1}+\frac{5^3}{e^{5\pi}+1}+\ldots\ldots\ldots$$

$$=\frac{1}{48}+12\left\{\frac{1}{e^{2\pi}-1}+\frac{2^3}{e^{4\pi}-1}+\frac{3^3}{e^{6\pi}-1}+\ldots\ldots\right\}\ldots \quad (20)$$

§ 8. Equating the imaginary parts in the various equations used above we obtain several neat series; they are dealt with in the next part.

APPENDIX.

Values of q.	Corresponding values of		
	$Q_1 = \prod_1^\infty (1+q^{2n})$	$Q_2 \equiv \prod_1^\infty (1+q^{2n-1})$	$Q_3 \equiv \prod_1^\infty (1-q^{2n-1})$
$e^{-\pi}$	$2^{-\frac{1}{24}} e^{\frac{\pi}{12}}$	$2^{\frac{1}{4}} e^{-\frac{\pi}{24}}$	$2^{\frac{1}{8}} e^{-\frac{\pi}{24}}$
$e^{-\pi\sqrt{2}}$	$(\sqrt{2}+1)^{-\frac{1}{8}} . 2^{-\frac{3}{8}} . e^{\frac{\pi\sqrt{2}}{12}}$	$(\sqrt{2}+1)^{\frac{1}{8}} . 2^{\frac{1}{8}} . e^{-\frac{\pi\sqrt{2}}{24}}$	$2^{\frac{1}{4}} e^{-\frac{\pi\sqrt{2}}{24}}$
$e^{-\pi\sqrt{3}}$	$(\sqrt{3}+1)^{-\frac{1}{4}} . 2^{-\frac{7}{24}} . e^{\frac{\pi\sqrt{3}}{12}}$	$2^{\frac{1}{3}} . e^{-\frac{\pi\sqrt{3}}{24}}$	$(\sqrt{3}+1)^{\frac{1}{4}} . 2^{-\frac{1}{24}} . e^{-\frac{\pi\sqrt{3}}{24}}$
$e^{-\pi\sqrt{7}}$	$(3+\sqrt{7})^{-\frac{1}{4}} . 2^{-\frac{3}{8}} . e^{\frac{\pi\sqrt{7}}{12}}$	$2^{\frac{1}{2}} e^{-\frac{\pi\sqrt{7}}{24}}$	$(3+\sqrt{7})^{\frac{1}{4}} 2^{-\frac{1}{8}} e^{-\frac{\pi\sqrt{7}}{24}}$
$e^{-\frac{\pi}{\sqrt{2}}}$	$(\sqrt{2}+1)^{-\frac{1}{8}} . 2^{-\frac{3}{8}} . e^{\frac{\pi}{12\sqrt{2}}}$	$(\sqrt{2}+1)^{\frac{1}{8}} . 2^{\frac{1}{8}} . e^{-\frac{\pi}{24\sqrt{2}}}$	$2^{\frac{1}{4}} e^{-\frac{\pi}{24\sqrt{2}}}$
$e^{-\frac{\pi}{\sqrt{3}}}$	$(\sqrt{3}+1)^{-\frac{1}{4}} . 2^{-\frac{7}{24}} . e^{\frac{\pi}{12\sqrt{3}}}$	$2^{\frac{1}{3}} . e^{-\frac{\pi}{24\sqrt{3}}}$	$(\sqrt{3}+1)^{\frac{1}{4}} . 2^{-\frac{1}{24}} . e^{-\frac{\pi}{24\sqrt{3}}}$
$e^{-\frac{\pi}{\sqrt{7}}}$	$(3+\sqrt{7})^{-\frac{1}{4}} . 2^{-\frac{3}{8}} . e^{\frac{\pi}{12\sqrt{7}}}$	$2^{\frac{1}{2}} e^{-\frac{\pi}{24\sqrt{7}}}$	$(3+\sqrt{7})^{\frac{1}{4}} . 2^{-\frac{1}{8}} . e^{-\frac{\pi}{24\sqrt{7}}}$

On the Collinearity of three Points on a non-singular Cubic.*

By B. S. Madhavarao, M.Sc. and M. Lakshmanamurthi, M.A.

1. Introductory Remarks.

Every non-singular cubic can be reduced to the canonical form

$$x^3 + y^3 + z^3 + 6mxyz = 0;$$

and if three points (x_i, y_i, z_i) $(i = 1, 2, 3)$ on this cubic be collinear, we have the well-known condition due to Cayley†, *viz.*

$$x_1x_2x_3 + y_1y_2y_3 + z_1z_2z_3 = 0.$$

We have shown here that this condition is necessary but not sufficient.‡ We have further deduced other necessary conditions and we have been able to enunciate the following theorem :—

THEOREM:— *When three points on a non-singular cubic in the canonical form are represented in homogeneous co-ordinates* (x, y, z), *there cannot be only one necessary and sufficient condition of collinearity which is a rational integral function of the co-ordinates and symmetrical in* x, y, z.

We have also exhibited sets of conditions necessary as well as sufficient and shown that they must be at least two in number.

2. The General Problem.

We first proceed to show by *a priori* reasoning that there must be a single such condition for collinearity. In fact, we can choose any three points on a cubic in a 3-fold infinity of ways whereas there are only a 2-fold infinity of straight lines in the plane. Thus the imposing of a single condition is necessary and moreover ought to be sufficient for the collinearity of the three points on the cubic. This fact about the existence of a

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* A paper read before the South Indian Science Association, Bangalore, on 30th Sep 1923.

† See Cayley : *Collected Mathematical Papers*, Vol. 2 : p. 404.

‡ See Ganguli : *Higher Plane Curves ;* Vol 2, where it is stated that this condition is necessary as well as sufficient.

single condition has also been proved from other points of view. Representing the co-ordinates of any point on

$$x^3 + y^3 + z^3 + 6mxyz = 0$$

rationally in terms of the elliptic functions of a parameter u, Clebsch* has shown that, if u_1, u_2, u_3 be the parameters corresponding to the three points the *necessary* and *sufficient* condition for collinearity is

$$u_1 + u_2 + u_3 \equiv 0 \text{ (mod. } 2\omega, 2\omega')$$

or

$$u_1 + u_2 + u_3 \equiv 0 \text{ (mod. } 2k, 2k'i)$$

according as the Elliptic Functions are either Weierstrassian or Jacobian.

Let us consider the same question when the co-ordinates of the three points are simply taken as (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3). The general condition of collinearity for any three points in a plane is given by

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$

and the condition we are searching for is the form to which this reduces, when we make use of the facts that

$$x_1^3 + y_1^3 + z_1^3 + 6mx_1y_1z_1 = 0,$$
$$x_2^3 + y_2^3 + z_2^3 + 6mx_2y_2z_2 = 0,$$
$$x_3^3 + y_3^3 + z_3^3 + 6mx_3y_3z_3 = 0.$$

We can therefore solve for z_1, z_2, z_3 from each of the three equations above; substituting their values in the determinant, we obtain a relation among $(x_1, y_1, x_2, y_2, x_3, y_3)$ which is an algebraic function in them since the expression of z in terms of x and y from

$$x^3 + y^3 + z^3 + 6mxyz = 0$$

would involve only square root and cube root. We can thus deduce† a single necessary and sufficient condition even when the points are taken as (x_i, y_i, z_i); but it will not be a rational integral function of the co-ordinates and symmetrical in (x, y, z), and the actual labour involved in obtaining it is very great and, perhaps, not worth the trouble.

What we are concerned with in this paper is a condition which must involve x, y, z symmetrically and be also algebraic in them and the conclusion arrived at is embodied in the theorem of § 1.

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* See Clebsch : *Lecons sur la Géométrie* : *t*.2 (Fr. Edition).

† This point of view is pointed out to us by Prof. Hilton.

3. Cayley's Condition.

If the three points (x_i, y_i, z_i) $(i = 1, 2, 3)$ be collinear, we have the necessary condition due to Cayley, *viz.*

$$x_1x_2x_3 + y_1y_2y_3 + z_1z_2z_3 = 0.$$

In virtue of the importance of this result we just add an easy method of verification ;* consider the cubic

$$x^3 + y^3 + z^3 + 3kxyz = 0.$$

Any straight line $\dfrac{x-a}{l} = \dfrac{y-b}{m} = \dfrac{z-c}{n} = r,$

will meet it in points given by

$$\varphi(r) = \Sigma(a + lr)^3 + 3k\,\Pi(a + lr)$$

i.e., $$Ar^3 + 3Br^2 + 3Cr + D = 0,$$

where
$$A = l^3 + m^3 + n^3 + 3klmn$$
$$B = \Sigma(al^2) + k\,\Sigma(amn)$$
$$C = \Sigma(a^2l) + k\,\Sigma(abn)$$
$$D = a^3 + b^3 + c^3 + 3\,kabc.$$

Now
$$x_1x_2x_3 = (a + lr_1)(a + lr_2)(a + lr_3)$$
$$= \frac{(an - cl)^3 - (bl - am)^3}{A}$$

$\therefore$ $$\Sigma x_1x_2x_3 = 0,$$

which is Cayley's result.

It is quite easy to show that Cayley's condition is not sufficient. That is, if $$\Sigma x_1x_2x_3 = 0,$$ it does not necessarily follow that the three points are collinear. In fact, consider the line $xx_1x_2 + yy_1y_2 + zz_1z_2 = 0$ which passes through (x_3, y_3, z_3) if we suppose Cayley's condition to be satisfied. This intersects the cubic in two points other than (x_3, y_3, z_3) which are not collinear with (x_1, y_1, z_1) and (x_2, y_2, z_2) but yet satisfy the condition in question. Thus the condition although necessary is not sufficient and we can even specify that the order of insufficiency is two, inasmuch as there are two points which satisfy the condition but do not ensure collinearity.

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* Kindly suggested by Prof. M T. Naraniengar.

It follows therefore that the full geometrical import of Cayley's condition must be wider in scope than mere collinearity. We derive the geometrical significance of the condition from the following considerations. We know that if P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are two points in the plane of the cubic

$$x^3 + y^3 + z^3 + 6mxyz = 0 \quad \ldots \quad \ldots \quad \text{(i)}$$

the equation of the apolar line of P and Q with respect to the cubic is

$$x\,x_1\,x_2 + y\,y_1\,y_2 + z\,z_1\,z_2 + 2m\{\,x\,(y_1\,z_2 + y_2\,z_1) + \ldots\,\} = 0. \text{ (ii)}$$

Now if R (x_3, y_3, z_3) is any point on (ii) the relation between the co-ordinates is symmetrical and the triangle PQR is apolar to the cubic.

When P and Q are fixed points, we have an apolar line associated with each cubic of the syzygetic family (i), the apolar lines forming a pencil. When P and Q are points on a syzygetic cubic, the vertex of the pencil is the third point where PQ meets that cubic again.

Let us consider P and Q, two points on the cubic (i). The apolar line of P and Q with respect to the equi-anharmonic cubic

$$x^3 + y^3 + z^3 = 0$$

is

$$x_1\,x_2\,x + y_1\,y_2\,y + z_1\,z_2\,z = 0; \quad \ldots \quad \text{(iii)}$$

the triangle PQR where R is any point on (iii) is apolar to $x^3 + y^3 + z^3 = 0$. The line (iii) meets the cubic (i) in three points, one of which is the vertex of the apolar lines and is therefore the point where PQ meets the cubic (i) again. Hence the triangle formed by P, Q and any one of these points is apolar with respect to $x^3 + y^3 + z^3 = 0$. The geometrical significance of three points on a cubic satisfying Cayley's condition is therefore that "*the triangle formed by these points is apolar with respect to the equi-anharmonic cubic of the first species with its Hessian consisting of three real straight lines.*"

The property is an invariant one and as such, if it holds for the cubic (i) it also holds for the cubic in the general form.

4. Other linear conditions.

In addition to Cayley's condition, we have also a condition due to Hilton* *viz.*,

$$x_3\,(y_1z_2 + y_2z_1) + y_3\,(z_1x_2 + z_2x_1) + z_3\,(x_1y_2 + x_2y_1) = 0,$$

* See Hilton : *Plane Algebraic Curves*, p. 238 ; Ex. 13. We give a proof of this in § 5.

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which is also a necessary condition but not a sufficient one. This fact can be established by considering the intersections of the cubic by the line

$$x(y_1z_2 + y_2z_1) + \ldots\ldots + \ldots\ldots = 0;$$

we find two points in addition to (x_3, y_3, z_3) which satisfy the condition but are not collinear with (x_1, y_1, z_1) and (x_2, y_2, z_2).

We now proceed to establish a theorem, *viz.* :

THEOREM: *Cayley's and Hilton's conditions taken together are necessary and sufficient for the collinearity of the three points.*

For, if the join of A (x_1, y_1, z_1) and B (x_2, y_2, z_2) be not collinear with C (x_3, y_3, z_3) let it meet the cubic again at a point C′ (x_3', y_3', z'_3). Since A,B,C′ are collinear we have the two conditions

$$x_1x_2x'_3 + y_1y_2y'_3 + z_1z_2z'_3 = 0$$
$$(y_1z_2 + y_2z_1)\,x'_3 + \quad \ldots \quad + \quad \ldots \quad = 0$$

but since C also satisfies both the conditions we deduce that C and C- both lie on the lines

$$xx_1x_2 + \ldots\ldots + \ldots\ldots = 0;$$
$$(y_1z_2 + y_2z_1)x + \qquad\qquad = 0;$$

i.e. C and C′ are one and the same point. Thus the two conditions are together necessary and sufficient.

It is to be observed that the conditions due to Hilton and Cayley are linear in any one set of letters (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3). We now proceed to determine other such conditions which can be called *linear conditions.* We have the general condition of collinearity of any three points in the plane

$$x_3\,(y_1z_2 - y_2z_1) + \ldots\ldots + \ldots\ldots = 0$$

which is certainly a necessary condition. Combining this with Hilton's condition, *viz.*,

$$x_3\,(y_1z_2 + y_2z_1) + \ldots\ldots + \ldots\ldots = 0$$

which is also a necessary condition, we obtain two other necessary conditions by addition and subtraction in the form

$$x_3y_1z_2 + y_3z_1x_2 + z_3x_1y_2 = 0,$$
$$x_3y_2z_1 + y_3z_2x_1 + z_3x_2y_1 = 0;$$

while each of these conditions taken separately is merely necessary, we can establish, by a proof analogous to that used above that these two taken together are necessary as well as sufficient. These two conditions are further worthy of notice inasmuch as they give us remarkably simple expressions for the co-ordinates of the collinear point on the cubic. In fact, we have from the two equations above*

$$\frac{x_3}{(y_1z_1x_2^2 - y_2z_2x_1^2)} = \frac{y_3}{(z_1x_1y_2^2 - z_2x_2y_1^2)} = \frac{z_3}{(x_1y_1z_2^2 - x_2y_2z_1^2)}.$$

We can go still further and state the elegant result that of the three conditions

$$\left.\begin{aligned} x_1x_2x_3 + y_1y_2y_3 + z_1z_2z_3 &= 0 \\ y_1z_2x_3 + y_3z_1x_2 + y_2z_3x_1 &= 0 \\ x_3y_2z_1 + y_3z_2x_1 + z_3x_2y_1 &= 0 \end{aligned}\right\}$$

any two taken together constitute the necessary and sufficient conditions for the collinearity of the three points in question.

5. Necessary conditions of higher degree.

We have hitherto considered only conditions which are linear in terms of the co-ordinates of any one of the three points. We next proceed to determine other necessary conditions which are of higher degree, incidentally indicating a general method of deducing necessary conditions.

Let the cubic

$$x^3 + y^3 + z^3 + 6mxyz = 0$$

be cut by the line

$$\alpha x + \beta y + \gamma z = 0$$

in the points (x_1, y_1, z_1); (x_2, y_2, z_2) and (x_3, y_3, z_3).

Let us join with these two another arbitrary relation

$$px + qy + rz = 0.$$

The result of the elimination† of $x : y : z$ from these three expressions is

$$(\beta r - \gamma q)^3 + (\gamma p - \alpha r)^3 + (\alpha q - \beta p)^3 + 6m(\beta r - \gamma q)(\gamma p - \alpha r)(\alpha q - \beta p) = 0,$$

---

* These are the expressions for the collineal, otherwise deduced by Sylvester in his *Collected Mathematical Papers*: Vol. 3, pp. 354-5.

† This method of elimination is due to **Schlafi**: See his memoir "*Ueber die Resultante eines Systemes meherer algebraischer Gleichungen*" which has been expounded by **Cayley**. See his *Coll. Papers*, Vol. 2, pp. 454—64.

which is a ternary cubic in (p, q, r), say

$$(A, B, C, F, G, H, I, J, K, L)(p, q, r)^3 = 0,$$

where

$$A = \beta^3 - \gamma^3;\ B = \gamma^3 - \alpha^3;\ C = \alpha^3 - \beta^3$$

$$F = \gamma(\beta\gamma + 2k\alpha^2);\quad -I = \beta(\beta\gamma + 2k\alpha^2);$$

$$G = \alpha(\gamma\alpha + 2k\beta^2);\quad -J = \gamma(\gamma\alpha + 2k\beta^2);$$

$$H = \beta(\alpha\beta + 2k\gamma^2);\quad -K = \alpha(\alpha\beta + 2k\gamma^2);$$

and $$L = 0.$$

But this resultant can also be put in the form

$$(px_1 + qy_1 + rz_1)(px_2 + qy_2 + rz_2)(px_3 + qy_3 + rz_3) = 0,$$

so that we have the relations which express A, B and C in terms of the fundamental symmetrical functions of of the system of roots $(x_1\ y_1\ z_1)$, *viz.*,

$$A = x_1 x_2 x_3$$

$$B = y_1 y_2 y_3$$

$$C = z_1 z_2 z_3.$$

$$3F = y_1 y_2 z_3 + y_2 y_3 z_1 + y_3 y_1 z_2;$$

$$3I = y_1 z_2 z_3 + y_2 z_3 z_3 + y_3 z_2 z_2$$

$$3G = z_1 z_2 x_3 + z_2 z_3 x_1 + z_3 z_1 x_2;$$

$$3J = z_1 x_2 x_3 + z_2 x_3 x_1 + z_3 x_1 x_2$$

$$3H = x_1 x_2 y_3 + x_2 x_3 y_1 + x_3 x_1 y_2;$$

$$3K = x_1 y_2 y_3 + x_2 y_3 y_1 + x_3 y_1 y_2$$

$$6L = \Sigma x_3(y_1 z_2 + y_2 z_1).$$

Thus A, B, C, ... L are expressed in terms of (x_i, y_i, z_i) as well as of (α, β, γ) and by eliminating $\alpha : \beta : \gamma$ between any three expressions thus obtained, we shall have a necessary condition involving (x_i, y_i, z_i).

We observe immediately that

$$6L = \Sigma x_3(y_1 z_2 + y_2 z_1) = 0,$$

which is no other than Hilton's condition.

Further $$A + B + C = \Sigma(\beta^3 - \gamma^3) = 0.$$

$$\therefore\quad \Sigma x_1 x_2 x_3 = 0$$

which is Cayley's condition.

We have further, since

$$F = \gamma(\beta\gamma + 2k\alpha^2) \text{ and } -I = \beta(\beta\gamma + 2k\alpha^2),$$

$$\frac{F}{I} = -\frac{\gamma}{\beta}.$$

Similarly $$\frac{G}{J} = -\frac{\alpha}{\gamma},$$

and $$\frac{H}{K} = -\frac{\beta}{\alpha};$$

so that $$\frac{F}{I}\cdot\frac{G}{J}\cdot\frac{H}{K} = -1,$$

i.e., $$FGH + IJK = 0.$$

This is therefore a necessary condition which is of degree three in the co-ordinates of any one set, since F, G, H, I, J, K are all linear by themselves. We can similarly deduce conditions of degrees six, nine, etc., but we shall take

$$FGH + IJK = 0,$$

as typical of the higher degree conditions. That this is a necessary condition is obvious from its very method of derivation. We now proceed to show that it is really a condition independent of the linear conditions deduced in § 5 and that it is not also a sufficient condition. The condition can be written in the form

$$27(FGH + IJK) = 0:$$

and putting $$l = x_1 x_2;\ m = y_1 y_2;\ n = z_1 z_2$$

$$l' = y_1 z_2 + y_2 z_1;\ m' = z_1 x_2 + z_2 x_1;\ n' = x_1 y_2 + x_2 y_1$$

this can be written in the form

$$\begin{aligned} &2(lmn + l'm'n')x_3y_3z_3 + m(m'n' + ll')z_3^2x_3 \\ &+ m(l'm' + nn')z_3x_3^2 + n(n'l' + mm')x_3^2y_3 \\ &+ n(m'n' + ll')x_3y_3^2 + l(l'm' + nn')y_3^2z_3 \\ &\qquad + l(n'l' + mm')y_3z_3^2 = 0. \end{aligned}$$

Using the conditions

$$A + B + C = 0,\quad L = 0$$

i.e., $$lx_3 + my_3 + nz_3 = 0$$

and $$l'x_3 + m'y_3 + n'z_3 = 0$$

this reduces on rearrangement to the symmetrical form

$$\begin{aligned} &lmn'z_3^3 + mnl'x_3^3 + nlm'y_3^3 \\ &\quad + (mn'^2 + nn'^2 + ll'^2 - 2lmn - 2l'm'n')x_3y_3z_3 = 0. \end{aligned}$$

Making x_3, y_3, z_3 the current co-ordinates, this represents a non-singular cubic and hence the condition cannot reduce to the cube of a linear expression as otherwise the corresponding cubic would have a triple point.

Thus the condition

$$\mathrm{FGH} + \mathrm{IJK} = 0$$

is really a condition independent of the linear conditions. This same conclusion can also be reached by examining the cubic

$$\Sigma mnl'x^3 + \{ \Sigma ll'^2 - 2lmn - 2l'm'n' \} xyz = 0$$

for a triple point at (x_3, y_3, z_3), *i.e.*, taking Sylvester's form for the collineal at

$$(y_1z_1x_2^2 - y_2z_2x_1^2, \ldots\ldots, \ldots\ldots).$$

That this condition is insufficient follows very easily. The cubic considered above intersects the original cubic in nine points of which (x_3, y_3, z_3) alone is collinear with (x_1, y_1, z_1) and (x_2, y_2, z_2), whereas the other eight points while they satisfy the condition

$$\mathrm{FGH} + \mathrm{IJK} = 0$$

are non-collinear with them. We can thus state that the order of insufficiency is 8 in this case. Similarly any condition of a higher degree can be proved to be insufficient. In fact, we can state that for a condition

$$f_n(x\ y_i\ z_i)_{i=1,2,3} = 0$$

the order of insufficiency is $3n - 1$; for making (x_3, y_3, z_3) the current co-ordinates the equation would represent an *n-ic* which intersects the original cubic in $3n$-points of which (x_3, y_3, z_3) is one. Thus any one of the remaining $(3n - 1)$ points taken with the points (x_1, y_1, z_1) and (x_2, y_2, z_2) would satisfy the above condition but would not be collinear with those points.

Definition of Number*.

By S. V. Ramamurty, M.A., I.C.S.

1. In his "*Introduction to Mathematical Philosophy*," Bertrand Russell defines number as the class of similar classes. This definition was first given by Frege in 1884 but was practically ignored till it was rediscovered by Russell. I am not aware of any other definition of number by a mathematician.

Taking a particular number, 2 is defined as the class of couples. The class (or collection) of Mr. and Mrs. A, the class of Mr. and Mrs. B, the class of Mr. and Mrs. C and so on are similar classes as there is a one-to-one relationship between members of any two classes. The class of these similar classes is the number 2. As criticised by Professor Alexander in his "*Space, Time and Deity*," this definition tells us little about what number is—as little as saying that man is the class of men tells us about men—and is a definition by extension. Russell himself points out that a definition by intension is logically more fundamental than one by extension. It is indeed permissible to hold with Alexander that the similarity of classes is itself the result of each possessing a common characteristic, *viz.*, a common number. Take a couple. The number 2 is already involved in that collection. It is immaterial whether in the world there are other couples. But the existence of other couples is necessary for defining 2 as Russell does. I submit that Russell's analysis of the notion of number is not deep enough.

2. Professor Alexander seeks to give a definition of number by intension by stating it to be the plan of a whole of parts. This definition is not definite enough. There is not necessarily only one relation between a whole and its parts. A number may be one scheme of relationship but not the only one. Take two men standing together. The plan of the whole of that collection of parts may be stated in different ways. We may say the whole is formed by the men standing back to back. This definition tells us as little as to what number is as Russell's.

I consider that the notion of number needs to be analysed further so as to yield a definition which is deeper than Russell's and more definite than Alexander's.

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* Presented to the Fourth Session of the Indian Mathematical Conference, Poona, April 1924.

3. In my paper on "*Time, Space, Matter and Mind*" (Vol. XV, No. 1 of this Journal) I quoted from Bergson's *Introduction to Metaphysics* that "quantity is incipient quality" and used that dictum in building up my general hypothesis as to the relation of time, space, matter and mind. But what is at the basis of quantity itself? I regard the world as evolved out of the undifferentiable primeval stuff into time, then space, then matter and then mind. Take the first evolution of an atom of time—the smallest particle of time that we cognize. Any aggregation less than that does not develop the quality of time and is not cognized as time. The emanation of that quality involves the emanation of an incipient quantity. We can say that quality is incipient quantity. Imagine that we watch the first creation of the world. An atom of time A forms. Then an atom of time B forms. Then an atom of time C. We call each an atom of time not by comparing each of them with one of them, but by comparing each atom of time with the notion of timeness in our mind.

Compare a collection of horses. Let us watch the aggregate grow and imagine that first the legs and then the head and then the full body of a horse appear. In the course of the growth, there is no horse when there are only legs, none when there are legs and the head only and so on. A point in the growth is reached when a horse evolves and we cognize it by comparing the evolved product with the notion of horseness in our mind. So too again we say there is another horse when again the evolved product agrees with the notion of horseness in our mind. The two horses may be one big and one small, one an Arab and one a Waler. But as against the notion of horseness, we attach the number 2 to that collection. So too a collection of a man and a dog is a collection of 2 animals—the quality evolved in each instance being animalness. A horse and a book are a collection of 2 things—each evolving the quality of thingness. Thus number is attached to an aggregate of perceptions of quality irrespective of the variety of the quality. The quality of time is the primary quality evolved in the Universe. It is so on the Einsteinian theory where all entities are made of space-time and space is made up of three independent directions any one of which may furnish a time line.

Take the number 2; it is attached to 2 men, a man and a dog, a horse and a book and also 2 atoms of time. Of these classes, the class of 2 atoms of time is the most primary class, as time is the most primary of all entities. Number is independent of the particular quality of the entities. All that it needs is that there should be an entity. The least an

entity can be is to be an atom of time. Hence the number 2 (and so too any other natural number) is attached to a piece of time just as it is attached to a group of animals. What is the mode of this attachment ?

Russell begins by saying that most philosophers have fallen into the error of confusing a trio of men for an instance of the number 3 (*vide* Chapter II of his book on *Mathematical Philosophy*, already quoted). Having defined number as a class of classes, he states that we naturally think that the class of couples is something different from the number 2; but adds :

" There is no doubt about the class of couples. It is indubitable and not difficult to define whereas the number 2, in any other sense, is a metaphysical entity about which we can never feel sure that it exists or that we have tracked it down. It is therefore more prudent to content ourselves with the class of couples, which we are sure of, than to hunt for a problematical number 2 which must always remain elusive."

Thus Russell while denying that number is a class is content to say it is a class—of classes—and this on the ground of prudence.

Further, as already pointed out, to call number a class of (similar) classes gives no information as to what it is.

The same arguments that Russell gives for identifying a number with a class of similar classes apply to identifying number with a class, provided we can find a suitable class. Such a class I submit is an aggregate of time atoms. All entities evolve from time atoms. A number of time atoms is involved in the same number of any other entities. If we say the number is that aggregate of time, it is involved in the other aggregates also and hence is attached to them too. There is no need either in logic or from prudence to regard 2 as separate from the piece of time to which it is attached. I define a number or rather, I define a natural number, with which alone we have been dealing, thus:—

" A number is a piece of time."

Compare: Russell's definition : 'A number is a class of similar classes;' and Alexander's definition : 'A number is the plan of a whole of parts.'

As against Russell's, I regard the class of time atoms to be the most primary of the similar classes; and as against Alexander's, I may say that a piece of time is a whole of parts and its plan is involved in wholes of more complex entities.

4. Let M be a natural number; that is, a piece of time.

We define M in M horses as a relational number by stating that M horses have a one-to-one relationship with M.

Russell treats + M, -- M, M/N, iM as relational numbers when M is a natural number. This view can be accepted on my definition too. I extend it to M in M horses. So also dinitive and trinitive numbers can be treated as relational numbers.

The primary identity

$$1 = 1$$

is true not because we identify 1 with 1, but both with the notion of one-ness in our mind. Hence the primary identity can be fully written out in the form

$$1 = 1, \text{ relatively to } 1;$$

or, in the form

$$1 = 1 \parallel 1.$$

Some new formulæ connected with the working of a Superannuation Fund

By M. Vaidyanathan, M.A., L.T.

[In the paper on "*An Actuarial Study of a Superannuation Fund for the Teachers of the Madras Presidency*" read at the Fourth Indian Mathematical Conference, the writer exhaustively dealt with the fundamental principles that govern the working of such funds, the methods of tabulation and graduation to be adopted, the commutation columns necessary for calculating the present values of future liabilities due to withdrawal, mortality and retirement, and lastly various points of purely actuarial interest arising out of valuation of such funds. The mathematical formulæ which form Part II of the paper are extracted below.]

In this paper, x denotes the age of entry, and s_x the average salary between ages x and $(x + 1)$.

Also l_x represents those in service at age x

d_x ,, dying in service between x and $(x + 1)$

w_x ,, withdrawing ,, ,,

r_x ,, retiring ,, ,,

Section A.

Present value of future contributions.

Assuming that contributions to the fund are a definite percentage of salaries, the present value of 1 per cent of future salaries

$$= 01 \int_x^w v^t \, l_{x+t} \, s_{x+t} \, dt$$

where w is the limiting age of the service, and l_{x+t} represents those in active service at the end of time t out of l_x persons entering service. This is on the assumption that l_x and s_x are continuous functions of x. But we shall suppose that deaths, withdrawals and retirements are uniformly distributed throughout the whole year and that the salaries are received at the middle of the year. Then the present value, equated to 1 of the present salary s_x, of every one of l_x persons

$$= \frac{\cdot 01}{l_x s_x} \left\{ v^{\frac{1}{2}} l_{x+\frac{1}{2}} s_x + v^{\frac{3}{2}} l_{x+\frac{3}{2}} s_{x+1} + \ldots\ldots\ldots \right\}$$

$$= \frac{\cdot 01}{v_x l_x s_x} \left\{ v^{x+\frac{1}{2}} l_{x+\frac{1}{2}} s_x + v^{x+\frac{3}{2}} l_{x+\frac{3}{2}} s_{x+1} + \ldots\ldots\ldots \right\}$$

where $l_{x+\frac{1}{2}}$ can be taken to be $\frac{1}{2}(l_x + l_{x+1})$, the summation being extended to the limiting age of service.

Now, let $D_x = v^x l_x$; $D^s_x = v^x l_x s_x$

$$D^{(2)s}_x = v^{x+\frac{1}{2}} l_{x+\frac{1}{2}} s_x; \sum_x^w D^{(2)s}_x = N^{(2)s}_x.$$

Then, present value = $\cdot 01 \left[N^{(2)s}_x / D^s_x \right]$.

Hence the tables necessary for the calculation of the present value of all future contributions are $l_{x+\frac{1}{2}}$, $D^{(2)}_x$, $D^{(2)s}_x$, D_x and D^s_x.

SECTION B.

Present value of the benefit due to mortality or withdrawal at a uniform rate.

If the benefit on death is to be a fixed percentage of the contributions, that is, of the salaries as well, then, on the supposition of the uniform distribution of deaths, on an average

d_x persons get a fixed percentage of $(\frac{1}{2}s_x)$ in the middle of the year x,

d_{x+1} ..... $(s_x + \frac{1}{2} s_{x+1})$... of the year $(x+1)$,

d_{x+2} ..... $(s_x + s_{x+1} + \frac{1}{2} s_{x+2})$ ... of the year $(x+2)$

and so on.

Hence the total present value equated to the denominator D^s_x of the benefit of 1 per cent of salaries paid up to death

$$= \frac{\cdot 01}{D^s_x} \left\{ \frac{1}{2} s_x d_x v^{x+\frac{1}{2}} + \left(s_x + \frac{1}{2} s_{x+1} \right) d_{x+1} v^{x+\frac{3}{2}} + \ldots\ldots \right\}$$

$$= \frac{\cdot 01}{D^s_x} \left\{ s_x \left(d_x v^{x+\frac{1}{2}} + d_{x+1} v^{x+\frac{3}{2}} + \ldots\ldots \right) \right.$$

$$+ s_{x+1} (d_{x+1} v^{x+\frac{3}{2}} + d_{x+2} v^{x+\frac{5}{2}} + \ldots\ldots) + \ldots\ldots$$

$$\left. - \frac{1}{2} (s_x d_x v^{x+\frac{1}{2}} + s_{x+1} d_{x+1} v^{x+\frac{3}{2}} + \ldots\ldots) \right\}.$$

$$\text{Assume} \quad {}_{x}v^{x+\frac{1}{2}} = \frac{{}_{x}v^{x+1}}{v^{\frac{1}{2}}} = \sqrt{1+i}\,{}^{d}\mathrm{C}_{x},$$

where 'd' signifies 'death' and

$$\sum_{x}^{w} \sqrt{1+i}\,{}^{d}\mathrm{C}_{x} = {}^{d}\mathrm{M}_{x};$$

also introducing salaries, let

$$\sqrt{1+i}\,{}^{d}\mathrm{C}_{x} \cdot s_{x} = {}^{d}\mathrm{M}'^{s}_{x};$$

$${}^{d}\mathrm{M}_{x} \cdot s_{x} = {}^{d}\mathrm{M}^{s}_{x}; \quad \sum_{x}^{w} {}^{d}\mathrm{M}^{s}_{x} = {}^{d}\mathrm{R}^{s}_{x}.$$

Then the required present value is equal to

$$\frac{.01}{\mathrm{D}^{s}_{x}} \left\{ {}^{d}\mathrm{R}^{s}_{x} - \tfrac{1}{2}\,{}^{d}\mathrm{M}'^{s}_{x} \right\}^{*} \qquad \ldots \qquad \ldots \ (b)$$

If a definite percentage of salaries be granted to withdrawals, then the present value of the benefit due to withdrawals is

$$\frac{.01}{\mathrm{D}^{s}_{x}} \left\{ {}^{w}\mathrm{R}^{s}_{x} - \tfrac{1}{2}\,{}^{w}\mathrm{M}'^{s}_{x} \right\}. \qquad \ldots \qquad \ldots \ (b')$$

where 'w' signifies 'withdrawal.'

SECTION C.

Present value of the benefit at varying rates.

In the previous section, a *uniform* percentage of contributions was considered as benefit to deaths or withdrawals. But if it increases by a constant difference at the ends of fixed periods of service, say, as per the following scheme :—

for the first 5 years of service, $k\%$ of salaries is returned ;

from the 6^{th} to the 10^{th}, $(k+l)\%$ ..... ;

from the 11^{th} to the 15^{th}, $(k+2l)\%$ .....

and so on, increasing by $l\%$ at the end of every quinquennium, then the total value of this benefit equated to the denominator D^{s}_{x} is

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* This formula, though identical with the one given by G. King (*vide* J. I. A., Vol. XLI) has the advantage of being free from too many notations.

$$\frac{{}^{\cdot}01}{D^{s}_{x}}\Big[k\Big\{\tfrac{1}{2}\,s_x\,d_x\,v^{x+\frac{1}{2}}+(s_x+\tfrac{1}{2}\,s_{x+1})\,d_{x+1}\,v^{x+\frac{3}{2}}+\ldots\text{to 5 terms}\Big\}$$

$$+(k+l)\Big\{(s_x+s_{x+1}+\ldots+\tfrac{1}{2}\,s_{x+5})\,d_{x+5}\,v^{x+\frac{11}{2}}+\ldots\text{to 5 terms}\Big\}$$

$$+(k+2l)\Big\{(s_x+s_{x+1}+\ldots+\tfrac{1}{2}s_{x+10})\,d_{x+10}\,v^{x+\frac{21}{2}}+\ldots\text{to 5 terms}\Big\}$$

$$+\text{etc.}\Big]$$

$$=\frac{{}^{\cdot}01}{D^{s}_{x}}\;[k\{s_x(M_x-M_{x+5})+s_{x+1}(M_{x+1}-M_{x+5})+\ldots\ldots\}$$

$$+(k+l)\{\sum_{x}^{5}s_x(M_{x+5}-M_{x+10})+s_{x+6}(M_{x+6}-M_{x+10})+\ldots\ldots\}$$

$$+(k+2l)\{\sum_{x}^{10}s_x(M_{x+10}-M_{x+15})+\ldots\ldots\}$$

$$-\tfrac{1}{2}\{k(\sum_{x}^{4}c_xs_x)+(k+l)\sum_{x+5}^{x+9}c_xs_x+\ldots\ldots\}],$$

$$=\frac{{}^{\cdot}01}{D^{s}_{x}}\;[kR^{s}_{x}+l\sum_{x+5}R^{s}_{x}+l\,\Sigma\,s_x\cdot\sum_{t=1}M_{x+5t}$$

$$-M_{x+5}\,\Sigma\,S_{x+5}-M_{x+10}\,\Sigma\,S_{x+10}-M_{x+15}\,\Sigma\,S_{x+15}-\text{etc}).$$

$$-\tfrac{1}{2}\{kM'^{s}_{x}+l\sum_{t=1}M'^{s}_{x+5t}\}],\qquad\ldots\qquad\ldots\ (c)$$

where $\sum_{x}^{m}s_x$ stands for $\sum_{x}^{x+m}s_x$

in every case the summations being taken to the limiting age of the table.

If we can tabulate $\Sigma\,M_{x+5t}$, $\Sigma\,M'^{s}_{x+5t}$ and $\Sigma\,R^{s}_{x+5t}$ for values of t and also $\sum_{x}^{m}S_x$, the evaluation of this formula becomes very simple. As we proceed by quinary intervals, the number of terms to be summed up gets small.

SECTION D.

Present value of the benefit at compound interest.

In the case of some funds, a definite percentage of contributions accumulated at compound interest is granted on death or withdrawal. For convenience, assume that the rate of valuation is the same as the rate of interest at which the benefit is calculated.

Now let $\Sigma\,d_x=l'_x$ which is different from l_x which is subject to three decrements of death, withdrawal and retirement.

Then $(l'_x - \frac{1}{2} d_x)$ persons will receive each in the middle of the first year a salary of s_x, whose present value

$$= v^{\frac{1}{2}}(l'_x - \tfrac{1}{2} d_x) \cdot s_x.$$

Similarly, $(l'_{x+1} - \frac{1}{2} d_{x+1})$ will receive each in the middle of the second year a salary of s_{x+1}, whose present value

$$= v^{\frac{3}{2}}(l'_{x+1} - \tfrac{1}{2} d_{x+1})\, s_{x+1};$$

and so on.

Hence the present value of 1 per cent. of the salaries accumulated at compound interest paid to those who die in the course of their service equated to 1 of the present salary s_x

$$= \frac{.01}{D^s_x} [v^{x+\frac{1}{2}}(l' - \tfrac{1}{2} d_x)\, s_x + v^{x+\frac{3}{2}}(l'_{x+1} - \tfrac{1}{2} d_{x+1}) s_{x+1} + \ldots\ldots]$$

Now, let

$$v^{x+\frac{1}{2}}(l'_x - \tfrac{1}{2} d_x) \cdot s_x = {}^{di}M^s_x$$

(this is King's notation with 'bar' over M omitted); also let

$$\Sigma\, {}^{di}M^s_x = {}^{di}R^s_x.$$

Then the present value

$$= \frac{.01}{D^s_x}\, {}^{di}R^s_x \quad \ldots \qquad \ldots \qquad \ldots (d)$$

A similar formula holds good in the case of withdrawals; but instead of the d_x column, we have to take up the w_x column.

Section E.

Present value of pensions.

Then we pass on to calculate the present value of pensions. We are not considering the pension as an annuity but as something guaranteed to a retiring employee over and above the contributions paid by him towards the fund. We allow retiring pensions, after a minimum of ten years service, due to serious disability. The pensions allowed are a definite percentage of the past salaries multiplied by the number of years of service (not exceeding 25). To get the present value of the future pensions :—

(1) We shall suppose that pensions are allowed even after 1 year of service, and they are 1 % of the past salaries multiplied by the number of years of completed service (without any maximum). Then on the hypo-

thesis of the uniform distribution of retirements throughout the year, the present value of the benefit due to retirements after the first year

$$= \cdot 01\ v^{\frac{5}{2}} . r_{x+1} (S_x + \tfrac{1}{2} S_{x+1}).$$

The value due to retirements after the second year

$$= \cdot 01\ v^{\frac{5}{2}} \times 2 \times r_{x+2} (S_x + S_{x+1} + \tfrac{1}{2} S_{x+2});$$

and so on.

Hence the total present value equated to the denominator D'_x.

$$= \frac{\cdot 01}{D'_x} \sqrt{1+i} \{ S_x (C_{x+1} + 2 . C_{x+2} + 3 . C_{x+3} + \ldots\ldots)$$
$$+ S_{x+1} (C_{x+1} + 2 . C_{x+2} + 3 . C_{x+3} + \ldots\ldots)$$
$$+ S_{x+2} (2 . C_{x+2} + 3 . C_{x+3} + 4 . x+4 + \ldots\ldots) + \text{etc.}$$
$$- \tfrac{1}{2} (C'_{x+1} + 2\, C'_{x+2} + 3\, C'_{x+3} + \ldots\ldots) \} .$$

But since

$$\sqrt{1+i} \{ C_{x+1} + 2 C_{x+2} + 3 C_{x+3} \ldots\ldots \} = R_x - M_x (= R_{x+1}),$$
$$\sqrt{1+i} \{ 2 . C_{x+2} + 3 . C_{x+3} + \ldots\ldots \} = R_{x+2} + M_{x+2},$$
$$\sqrt{1+i} \{ 3 . C_{x+3} + 4 . C_{x+4} + \ldots\ldots \} = R_{x+3} + 2 . M_{x+3},$$

and so on; also

$$\sqrt{1+i} \{ C'_{x+1} + 2\, C'_{x+2} + \ldots\ldots \} = R'_{x+1} \text{ (say)}$$

where $$R' = \Sigma M';$$

it follows that the total present value

$$= \frac{\cdot 01}{D'_x} \{ S_x (R_x - M_x) + S_{x+1} R_{x+1} + S_{x+2} (R_{x+2} + M_{x+2})$$
$$+ \ldots - \tfrac{1}{2} R'_{x+1} \}$$
$$= \frac{\cdot 01}{D'_x} \left[\Sigma R_x S_x - M'_x + \sum_0 t M'_{x+1+t} - \tfrac{1}{2} R'_{x+1} \right].$$

Assume $\Sigma R_x S_x = X'_x$; also $\sum_0 t\, M'_{x+1+t} = X'_{x+2} = X'_{x+1} - R'_{x+1}$

where $$X = \Sigma R = \Sigma (\Sigma M).$$

Hence the present value

$$= \frac{\cdot 01}{D'_x} \left[X'_x - M'_x + X'_{x+1} - R'_{x+1} - \tfrac{1}{2} R'_{x+1} \right]$$
$$= \frac{\cdot 01}{D'_x} \left[X'_x - R'_x + X'_{x+1} - \tfrac{1}{2} R'_{x+1} \right] \quad \ldots \qquad \ldots \quad (e)$$

Assume this to be $\frac{\cdot 01}{D_x^s}$. P_x^s.

(2) Now we shall consider the present value of the same benefit on the supposition that a minimum of 10 years' completed service is necessary for retirement. It is the same series as in (*e*) but whose first term is

$$\frac{\cdot 01}{D_x} \sqrt{1+i} \left\{ 10\, r_{x+10} v^{x+\frac{21}{2}} (S_x + S_{x+1} + \ldots + S_{x+9} + \tfrac{1}{2} S_{x+10}) \right\}.$$

Hence the present value

$$= \frac{\cdot 01}{D_x^s} \left\{ P^s_{x+9} + 9 \sum_x^8 s_x \left(\sqrt{1+i}\, \Sigma\, r_{x+10}\, v^{x+\frac{9}{2}}\right) \right\}.$$

$$= \frac{\cdot 01}{D_x^s} \left\{ P^s_{x+9} + 9 \sum_x^8 s_x \times M_{x-10} \right\}. \quad \ldots \quad \ldots \quad (e')$$

(3) Now we pass on to the present value of the same benefit where a minimum of 10 years completed service and a maximum of 25 years are fixed for pension. In this case even if the period of service exceeds 25, for purposes of pension the multiplying factor is taken to be 25. The present value is the same as (*e'*) diminished by

$$\frac{\cdot 01}{D_x^s} \left\{ P^s_{x+25} + \sum_x^{24} S_x \cdot \ (M_{x+26}) \right\}.$$

Hence the total present value

$$= \frac{\cdot 01}{D_x^s} \left\{ P^s_{x+9} - P^s_{x+25} + 9\, M_{x+10} \sum_x^8 S_x - M_{x+26} \sum_x^{24} S_x \right\}. \quad \ldots \quad (e'')$$

N.B.—At the inception of a fund, actuarial advice will be necessary to calculate the amount of pensions that could be granted consistently with the 'solvency' of the fund. Pension as an annuity has long ceased to be attractive, and there is clamour everywhere that it must be a definite lump sum at the time of retirement. Hence the formulæ worked out here; these when applied to the 'Life and Service' table give the actuary the exact pensions possible after allowing for benefits to withdrawals and mortality.

Variation of wind with height— Co-efficient of turbulence.*

By R. N. Apte.

[G. T. Taylor's recent investigation of the variation of wind with height (shortly stated: *geostrophic* wind) is taken as the standard of comparison, though in the equatorial belt the cyclostrophic terms cannot be neglected.

Taylor regards k the turbulence co-efficient as independent of height and obtains the relation for the surface wind S

$$S/G = \cos\alpha - \sin\alpha. \quad \ldots \quad \ldots (1)$$

which is amply supported by observations.

Harold Jeffreys is led to conclude that k the turbulence co-efficient depends upon the height. He proposes the law $k = k_0 (1+\lambda z)^2$ and suggests that (1) must be really

$$S/G = \cos\alpha - h \sin\alpha. \quad \ldots \quad \ldots (2)$$

As Taylor's equation (1) is amply supported by direct observational evidence and as k also is to depend upon height, any law proposed for the dependence of k upon height must be quite consistent with equation (1): the present writer thinks it useful to consider the law

$$k = k_0 (1 + \lambda z)^{2-\frac{2}{2n+1}}$$

for positive integral values, $n = 1, 2,$ etc., and proves that the results obtained are consistent with Taylor's equation (1).

An interesting case, $k = -k_0 z^{\frac{4}{3}}$ is examined.]

1. A great deal of light has recently been thrown upon the question of the variation of wind with height by G. T. Taylor's investigations of eddy-motion in the atmosphere. Taylor relies upon the *geostrophic wind* as representing the undisturbed wind in the upper air and, as remarked by Sir Napier Shaw, no other standard of reference seems possible. The *cyclostrophic term* is neglected though a consideration of it must be made with respect to the winds in the equatorial belt.

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2. As regards eddy-motion in the atmosphere, Taylor showed (*Phil. Transactions*, R. S., Vol. 215, p. 14) that if u and v be the components of velocity parallel to the horizontal axes of x and y, the rates of gain of momentum parallel to these axes, in unit of volume at height z are

$$k\rho\left(\frac{d^2u}{dz^2}\right),\quad k\rho\left(\frac{d^2v}{dz^2}\right), \qquad \ldots (1)$$

k being the constant used in temperature variations

$$\rho\sigma\frac{\delta\theta}{\delta t}=\frac{\delta}{\delta z}\left(k\rho\sigma\frac{\delta\theta}{\delta z}\right). \qquad \ldots \qquad \ldots \quad (2)$$

If the motion be steady the resultant gain or loss of momentum must balance the other forces acting on the element of volume considered. Thus k is a new meteorological quantity of great importance.

3. Taylor regards k as independent of height. Starting with the equations of motion of an incompressible viscous fluid given in Lamb's *Hydro-dynamics*, p. 338, he gets for horizontal motion equations equivalent to

$$2\omega v + k\frac{d^2u}{dz^2}=0 \qquad \ldots \qquad \ldots (3)$$

$$-2\omega u + k\frac{d^2v}{dz^2}=2\omega G \qquad \ldots \qquad \ldots (4)$$

where $\omega = \Omega \sin\lambda$, Ω is the earth's angular velocity of rotation, λ is the latitude and G the velocity of the geostrophic wind defined by

$$2\omega\rho G = -\frac{\partial p}{\partial y}.$$

The co-ordinate axes of x, y, z are such that z is measured vertically upwards and the axis is taken along the surface isobars. Now regarding k as independent of height in (3) and (4), he solves them obtaining

$$u = G - A_3 e^{-Bz}\cos Bz + A_4 e^{-Bz}\sin Bz \qquad \ldots (5)$$

$$v = A_3 e^{-Bz}\sin Bz - A_4 e^{-Bz}\cos Bz. \qquad \ldots (6)$$

The constants are found from boundary conditions and if S is the surface wind

$$S = G(\cos\alpha - \sin\alpha). \qquad \ldots \qquad \ldots (7)$$

Taylor's strong point is that this equation (7) is confirmed by observations and he holds that k, the co-efficient of turbulence, is independent of height.

4. Harold Jeffrey (*Proceedings* R. S. Vol. 96, p. 233) however, thinks that k is not independent of height. He has analysed a number of observations and examined the relations between S/G and α and the

possibility of a systematic contortion of isobars and is led to conclude that there must be a variation in the vertical distribution of turbulence.

He proposes the law $k = k_0 (1 + \lambda z)^2$ where k_0 and λ are constants and z is the height. He solves equations (3) and (4) with this hypothesis and gets

$$\frac{S}{G} = \cos \alpha - \sin \alpha \cot \beta, \qquad \ldots \qquad \ldots \quad (8)$$

where β is the argument of $\frac{m(m-1)}{m+1}$, and $W = G + A(1+\lambda z)^m$.

Here, however, it must be admitted that the real ratio is

$$\frac{S}{G} = \cos \alpha - h \sin \alpha \qquad \ldots \qquad \ldots \quad (9)$$

and that the mean value of h is about 1, so as to justify Taylor's equation (7) though the value of h varies over a wide range with individual observations.

As a rule, the wind determined by pilot baloon ascents, is found to agree fairly closely with the geostrophic wind at altitudes of 3000 feet and more. The question whether k is independent of height or not must be ultimately settled by observations which are not yet available. Direct quantitative evidence can be the complete test of the theory.

5. The present writer thinks it useful to try the laws

$$k = k_0 (1 + \lambda z)^{2 - \frac{2}{2n+1}}$$

where n is a positive integer. Harold Jeffrey's law is a particular case of this when n is ∞. The equations (3) and (4) combined into one single equation now give an equation in Ricatti's form which in this case is finite. But there would be a great difficulty in determining one of the constants.

6. We may, therefore, take successively

$$n = 1, 2, \&c. \ldots\ldots$$

Take the law $k = k_0 (1 + \lambda z)^{2 - \frac{1}{2}}$.

Equations (3) and (4) are

$$2\omega v + k \frac{d^2 u}{dz^2} = 0.$$

$$-2\omega u + k \frac{d^2 u}{dz^2} = 2\omega G ;$$

write $u + iv = w$. Then these equations combine into the equation

$$k \frac{d^2 w}{dz^2} - 2i\omega w = 2i\omega G.$$

$$\therefore \quad k\frac{d^2V}{dz^2} - 2i\omega V = 0, \text{ where } V = w - G.$$

$$\therefore \quad k\frac{d^2V}{dz^2} - (1+i)^2\omega V = 0. \quad \ldots \quad \ldots \quad (10)$$

Now $$k = k_0(1+\lambda z)^{\frac{1}{3}}.$$

$$\therefore \quad (1+\lambda z)^{\frac{1}{3}}\frac{d^2V}{dz^2} - c^2V = 0 \quad \ldots (11)$$

where $$c = \frac{(1+i)\omega^{\frac{1}{2}}}{\lambda k_0^{\frac{1}{2}}}; \quad \ldots \quad \ldots \quad \ldots (11')$$

put $$1 + \lambda z = \xi = t^3, \quad \ldots \quad \ldots (12)$$

and the solution of (11) is

$$w - G = V = A(1-3ct)e^{3ct} + B(1+3ct)\,e^{-3ct} \quad \ldots (13)$$

The constants A and B are to be determined from boundary conditions. Since the velocity shall not tend to infinity with height, A = 0.

$$\therefore \quad w = G + B(1+3ct)e^{-3ct}. \quad \ldots \quad \ldots (14)$$

If α denote the angle between the wind at the ground ($z = 0$) and the tangent to the isobar, we have at the surface

$$w = Se^{i\alpha}. \quad \ldots \quad \ldots (15)$$

$$\therefore \quad B = Se^{i\alpha} - G. \quad \ldots \quad \ldots (16)$$

Now another boundary condition is that the skin-friction at the surface is equal to $k_1\rho S^2$ per unit of area where k_1 is constant and that it acts in the direction opposite to the velocity. It must be equal to $k\rho$ times the rate of speed in the air near the surface.

$$\therefore \quad k_1\rho S^2 e^{i\alpha} = -\int_0^\infty k\rho\,\frac{d^2w}{dz^2}\,dz$$

$$= -3Bk_0\rho\left(c^2 + \frac{4c}{3} + \frac{8}{9} + \frac{8}{27c}\right)e^{-3c}, \quad \ldots (17)$$

taking the real part of c to be positive.

From (11′) writing $c = R(1+i)$ and putting

$$Be^{-i\alpha} = S - Ge^{-i\alpha} = Z_1, \quad e^{-3R(1+i)} = Z_2,$$

and $$\frac{2R}{3} + \frac{4}{9} + \frac{2}{27R} + i\left(R^2 + \frac{2R}{3} - \frac{2}{27R}\right) = Z_3,$$

where Z_1, Z_2, Z_3 are complex numbers, (17) may be written

$$k_1\rho S^2 = (\text{a real quantity}).\ Z_1Z_2Z_3. \quad \ldots \quad \ldots (18)$$

Let θ_1, θ_2, θ_3 be respectively the amplitudes of Z_1, Z_2, Z_3 thus Taylor's equation (7) will be true if

$$\tan \theta_1 = -1 \quad ie., \quad \frac{G \sin \alpha}{S - G \cos \alpha} = -1$$

resulting in $$S = G(\cos \alpha - \sin \alpha).$$

Now $$\tan \theta_2 = -\tan 3R$$

and $$\tan \theta_3 = \frac{27R^3 + 18R^2 - 2}{18R^2 + 12R + 2},$$

and subject to the condition mentioned above, we can adjust the constants k_0 and λ and therefore R, so that

$$\tan \theta_2 = \frac{1 - \tan \theta_3}{1 + \tan \theta_3}$$

or $$\tan 3R = \frac{27R^3 - 12R - 4}{3R(3R + 2)^2}; \quad \ldots \qquad \ldots (19)$$

and when this condition is fulfilled

$$\tan \theta_1 = -1.$$

Thus with this adjustment of the constants the law $k = k_0 (1 + \lambda z)^{\frac{4}{5}}$ satisfies Taylor's equation (7), which is confirmed by observations, and also the condition required by Harold Jeffrey that the co-efficient of turbulence k must depend upon height.

8. ·· It is to be noted that even if the value of $\tan \theta_1$ is not taken as -1 and modified by observations to be $-\tan \beta$, a known quantity, we can still adjust the constants k_0 and λ so that the equation (18) is true.

9. A similar treatment may be given to the law of § 5 when $n = 2$.

For equation (11) in this case, we have

$$(1 + \lambda z)^{\frac{8}{5}} \frac{d^2V}{dz^2} - c^2 V = 0 \qquad \ldots \qquad \ldots (20)$$

Put $1 + \lambda z = \xi = t^5$, and the solution is

$$w - G = V = A\left(1 - 3ct + \frac{25}{3} c^2 t^2\right) e^{5ct} + B\left(1 + 5ct + \frac{25}{3} c^2 t^2\right) e^{-5ct}$$

As before, from the condition that the velocity cannot tend to infinity with height, A must be equal to 0.

The condition of the skin-friction also may be similarly applied and we get an equation corresponding to (17).

10. It is interesting to note that if we take $k = -k_0 z^{\frac{4}{3}}$, equations (3) and (4) give

$$k_0 z^{\frac{4}{3}} \frac{d^2V}{dz^2} + 2i\omega V = 0,$$

which may be written as

$$k_0 z^{\frac{4}{3}} \frac{d^2V}{dz^2} - (1-i)^2 \omega V = 0. \quad \ldots \quad \ldots \quad \ldots (21)$$

Put $z = x^3$, the solution is $w = G + A(1-3cx)^{+3cx} + B(1+3cx).^{-3cx}$

As before A must be zero.

$$\therefore \quad w = G + B(1+3cx),^{-3cx} \quad \ldots \quad \ldots \quad \ldots (22)$$

where $x = z^{\frac{1}{3}}$ and $c = \dfrac{(1-i)\,\omega^{\frac{1}{2}}}{k_0^{\frac{1}{2}}}$.

Applying the condition of skin-friction as before, we get

$$k_0 S^2 e^{i\alpha} = \frac{8k_0}{3c} B = \frac{8k_0}{3c} (Se^{i\alpha} - G).$$

$$\therefore \quad k_1 S^2 c = 8k_0 (S - Ge^{-i\alpha})$$

and equating the ratios of the real and imaginary parts on the two sides, we get

$$\frac{G \sin \alpha}{S - G\cos\alpha} = -1.$$

$$\therefore \quad \frac{S}{G} = \cos\alpha - \sin\alpha,$$

which is exactly Taylor's equation; but a physical interpretation of this case is difficult as k becomes zero at the surface.

Invariant-Factors and Integer-Sequences.

By R. VAIDYANATHASWAMY, M.A., F.R.S.E.

The theory of characteristic (or 'latent') roots of a matrix is perfect smooth-sailing when they are all distinct. But when a characteristic root α is repeated a certain number of times—say r times, then a little reflection will suffice to show that the properties of the matrix in relation to the characteristic root α are *not* completely determined by a knowledge of the number r alone. That is to say, over and above the *fact* of coalescence of r characteristic roots, there is a *manner* of coalescence about which the number r by itself gives no information. To specify completely this 'manner' of coalescence, requires a sequence (generally a decreasing sequence) of integers.

There are two classical theories of the matrix, the *algebraic* theory of Weierstrass and the *geometric* theory of Grassman. In the former, the representative sequence of a multiple characteristic root α is the sequence of the indices of the *invariant-factors* corresponding to α : or briefly, the *invariant sequence* of α. In the latter, the matrix is regarded as the symbol of a collineation in hyperspace, and the representative sequence of α (which I propose to call the 'latency sequence') is derived from the orders of the latent and semi-latent regions of successive species for the 'latent' root α. The relation between the invariant sequence and the latency sequence is well-known. In this order of ideas, there arises naturally the problem of specifying the process by means of which one can derive from the representative sequences (that is to say, *either* the invariant *or* the latency sequences) of a common latent root α of two matrices Δ_1, Δ_2, the representative sequence of α with respect to the "direct product" of Δ_1 and Δ_2. It is clear that this process could be stated solely in terms of modes of combination of integer-sequences; that is to say, the matrix may be eliminated altogether, and the question may be treated as one on abstract decreasing sequences of positive integers. When this is done, the algebraic and the geometric theories will be put in very clear relation to one another.

We begin by treating of various relations and processes connected with *decreasing sequences of positive integers*. The results and ideas are then applied to the sequences which occur in the Weierstrassian and the Grassman theory.

§ 1. **Sequences.** By a 'sequence' we shall invariably understand (unless explicitly stated otherwise) a *decreasing sequence of positive integers*, that is, a sequence of the type

$$R = (r_0\ r_1 \dots r_n)$$

where the r's are positive integers such that $r_k \geqslant r_{k+1}$. We shall also call the sum $r_0 + r_1 + \dots\dots + r_n$, the *weight* of the sequence R.

We shall find it useful to think that the sequence R consists, in addition to the r's of an infinite number of zero terms added at the end. From this point of view, every sequence R consists really of an infinite number of terms, of which however only a finite number at the beginning are greater than zero; and it is only for convenience that R is *denoted* by its non-vanishing part.

§ 2. **The Sum and the Compound.**

The *sum* of two sequences

$$R = (r_0\ r_1 \dots r_n),$$
$$S = (s_0\ s_1 \dots s_m),$$

is defined to be the (decreasing) sequence

$$R + S = (t_0\ t_1\ t_2 \dots),$$

in which $t_k = r_k + s_k$ (r_k and s_k being considered to be zero, in accordance with our previous remark, when k is greater than n and m respectively).

The *compound* of the two sequences R and S is defined to be the sequence

$$R \oplus S = (t_0\ t_1 \dots t_{m+n+1}),$$

where the t's consist of the r's and s's arranged in descending order of magnitude.

The sum and the compound of any number of sequences are defined in the same manner. From the definition it is clear that the weight of the sum or the compound of any number of sequences is the sum of their weights. It is also clear that the operation of taking the sum or the compound is commutative and associative.

§ 3. **The Conjugate of a Sequence.**

A sequence $R = (r_0 r_1 \dots r_n)$ of weight w equal to $(r_0 + r_1 + \dots + r_n)$ may be regarded as representing a special kind of partition of the number w, namely a partition into decreasing parts. Thus the sequence R may be represented graphically by $n+1$ columns of dots, the kth column contain-

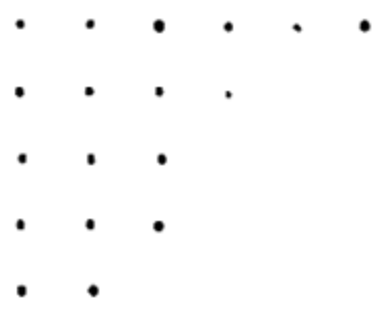
ing r_{k-1} dots and the top-dots of all the columns forming a horizontal row. If we turn this diagram through a right angle, we obtain the diagram of a new partition (called the *conjugate* of the original partition), also into decreasing parts of the same number w; and this new partition will correspond to a decreasing sequence S, which we define to be the *conjugate sequence* of R.

Two conjugate sequences R and S, where

$$R = (r_0\ r_1\ \dots\ r_n)$$
$$S = (s_0\ s_1\ \dots\ s_m),$$

are obviously related in this reciprocal manner, that $r_0 = m+1$, $s_0 = n+1$ and generally r_k = the number of s's which are greater than k, and s_k = the number of r's which are greater than k.

As an illustration, the sequence (5, 5, 4, 2, 1, 1) is represented by the graph



On turning the page through a right angle, the graph reads the partition corresponding to the *conjugate sequence* (6, 4, 3, 3, 2).

The three operations of taking the sum, the compound and the conjugate are connected by the following simple relation :—

THEOREM I. *The sum of the conjugates of any number of sequences is the conjugate of their compound*; or in symbols (denoting the conjugate of R by | R),

$$|R + |S + \dots = |(R \oplus S \oplus \dots).$$

The graphical proof of this theorem will become obvious by considering the case of two sequences A = (4, 3, 3, 2, 1, 1) and B = (6, 3, 2. 2).

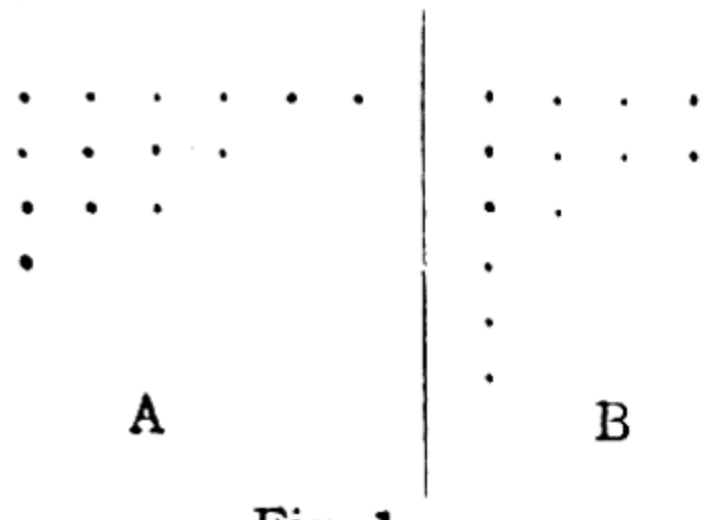


Fig. 1.

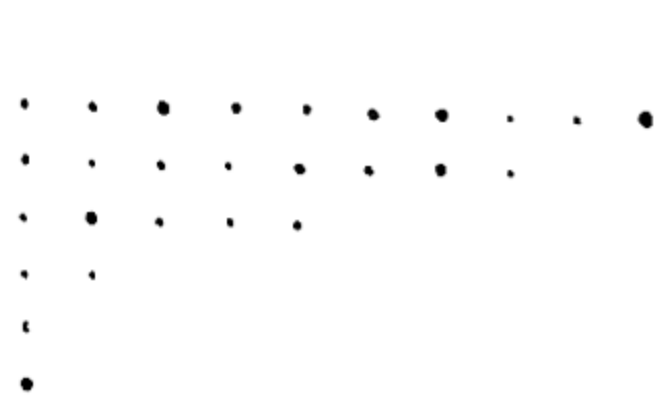


Fig. 2.

Place the graphs of A and B together as in Fig. 1. Then push every dot of B along the same horizontal line, to the left as far as it would go; we thus get Fig. 2.

It is obvious that Fig. 2 read by columns, gives the compound of the two sequences A and B. Now the sum of the conjugates of A and B is obtained by reading Fig. 1 by rows; but in obtaining Fig. 2 from Fig. 1, we have not altered the number of dots in any horizontal row. Hence the sum of the conjugates of A and B is also obtained by reading Fig. 2 by rows. Thus Fig. 2 read by columns gives the *compound*, and read by rows the *sum* of the conjugates of the two sequences, so that the conjugate of the compound is the sum of the conjugates.

§ 4. The Derivate and the Integral.

If $R = (r_0\ r_1 \cdots r_n)$ be a decreasing squence, the sequence

$$\Delta R = R' = (r_0'\ r_1' \ldots r_n')$$

where $$r_0' = r_0 - r_1,\ r_1' = r_1 - r_2,\ \ldots\ r_n' = r_n)$$

is a sequence of positive integers, but not necessarily a decreasing sequence If R' *is* a decreasing sequence, then R may be said to be *differentiable*, or to admit the *derivate* $\Delta R = R'$.

The *integral sequence* of R is defined to be the unique (differentiable) sequence the derivate of which is the sequence R. From the definition of the derivate, it follows that the integral of $R = (r_0\ r_1\ .\ .\ r_n)$ is given by

$$\Delta^{-1} R = (R_0\ R_1 \cdots R_n)$$

where $$R_k = r_k + r_{k+1} + \cdots + r_n\ (k = 0, 1 \cdots n).$$

It may be remarked that a sequence $R = (r_0)$ containing only one term is its own integral and derivate.

§ 5. The Product.

Let there be two sequences R and S given by

$$R = (r_0\ r_1 \cdots r_n)$$

$$S = (s_0\ s_1 \cdots s_m).$$

Consider the sequence $T = (t_0\ t_1 \ldots)$ defined by the relations:

$t_0 =$ $r_0 + s_0$

$t_1 =$ the smaller of the numbers $r_0 + s_1,\ r_1 + s_0$;

$t_2 =$ the smallest of the numbers $r_0 + s_2,\ r_1 + s_1,\ r_2 + s_0$;

and generally $t_k =$ the smallest of the numbers $(r_\lambda + s_{k-\lambda})\ (\lambda = 0, 1 \ldots k)$.

The sequence T thus defined is necessarily a decreasing sequence. To prove this, assume that the least of the numbers $(r_\lambda + s_{k-\lambda})$ $(\lambda = 0, 1, \ldots k)$, is given by the value p_k of λ, p_k being an unknown function of k, Then we have :

$$\begin{aligned} t_{k+1} &= r_{p_{k+1}} + s_{k+1-p_{k+1}} \text{ (by definition of } t_{k+1}) \\ &\leqslant r_{p_k} + s_{k+1-p_k} \text{ (by definition of } p_{k+1}) \\ &\leqslant r_{p_k} + s_{k-p_k} \text{ (since S is a decreasing sequence)} \\ &\leqslant t_k. \end{aligned}$$

Thus T is a decreasing sequence. *The sequence T obtained thus, is defined to be the product of R and S, and written* $R \times S$.

The product of any number of sequences is easily seen to be commutative and associative.

The number of terms in a product sequence is easily shewn to be the sum of the numbers of terms in the factor sequences.

An important relation between the compound and the product is given by the following theorem :—

THEOREM II. *The Integral of the compound of any number of sequences is the product of their integrals.* In symbols:

$$\Delta^{-1}(R \oplus S \oplus T \oplus \ldots) = \Delta^{-1}.R \times \Delta^{-1}.S \times \Delta^{-1} T \times \ldots$$

It will be sufficient to indicate the proof for two sequences $R = (r_0\ r_1 \ldots r_n)$, $S = (s_0\ s_1 \ldots s_m)$.

Clearly, the $(k+1)$th term of the integral of their compound is :

$\Sigma r + \Sigma s$ — the sum of the k greatest numbers of the set of the r's and the s's.

But, the sum of the k greatest numbers of the set of the r's and the s's is the greatest of the quantities which are the sums of the λ greatest r's and the $k-\lambda$ greatest s's $(\lambda = 0, 1 \ldots k)$. Thus, the $(k+1)$th term of the integral of the compound is the *least* of the quantities of the type :

$(\Sigma r$ — the sum of the λ greatest r's$) + (\Sigma s$ — the sum of the $k - \lambda$ greatest s's$)$ $(\lambda = 0, 1 \ldots k)$.

But this is precisely the $(k+1)$th term of the product of the integrals of the sequences R and S.

The theorem is thus proved.

As a special case, we have the result that the product of the one-term sequences (a), (b), (c) ..., where $a \geqslant b \geqslant c \cdots$, is the integral of the sequence $(a, b, c, \ldots)$.

Ex. Investigate the condition that a sequence R may be divisible by a sequence S.

§ 6 The Weierstrassian Theory of the Matrix.*

If M be a given matrix, the values of λ for which the determinant

$$| M - \lambda |$$

is equal to zero are the *characteristic* or the *latent roots* of M. If α is an r-ple characteristic root of M, then the determinant $| M - \lambda |$ would be divisible by $(\lambda - \alpha)^r$; it will happen further that all the kth minors of this determinant would be also divisible by a finite power of $\lambda - \alpha$, say $(\lambda - \alpha)^{r_k}$ $(k = 1, 2 \ldots)$. It may be shown that the sequence

$$R = (r_0\, r_1\, r_2 \ldots), \quad (r_0 = r)$$

(which may be called the *factor-sequence*) is not only a decreasing sequence, but a *differentiable* sequence; that is to say, the sequence

$$S = (s_0\, s_1 \ldots) \quad (s_0 = r_0 - r_1, \cdots s_k = r_k - r_{k+1}),$$

(which will be called the *invariant-sequence*) is also a decreasing sequence, and in fact $(\lambda - \alpha)^{s_k}$ $(k = 0, 1 \ldots)$ are the invariant factors of M with respect to the characteristic root α.

The essence of Weierstrass' method consists in showing that the factor-sequence R, or what comes to the same thing, the invariant sequence S, completely characterises the relation of M to its multiple characteristic root α; or, in other words, *the necessary and sufficient condition that it may be possible to transform two matrices M, M′ into one another, by a relation of the form*

$$M' = \Delta M \Delta' \quad (|\Delta| \neq 0, |\Delta'| \neq 0),$$

is the equality of their invariant-factors.

If M, M′ are interpreted as bilinear forms in two sets of variables, it would follow that the equality of the invariant factors is the condition for the transformability of the two forms into one another, by independent linear transformations of the two sets of variables. If M, M′ are considered as collineations between two spaces Σ, Σ', the equality of their invariant-factors gives the condition that M, M′ represent the *same* collineation under different co-ordinate systems.

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* For an exposition of this theory, see the Cambridge Tract on *Quadratic Forms* or *Encylopédie des Sciences Mathematiques*, Tome I (ii) Fas. 4, or Hilton's *Linear Substitutions*.

§ 7. Compartite Matrices.*

A matrix M is said to be a *compartite* matrix, if it is of the form :

$$M = \left|\begin{array}{c|c} \Delta' & \\ \hline & \Delta'' \end{array}\right|,$$

where Δ', Δ'' are two matrices, and the blank spaces are occupied by zero elements ; Δ', Δ'' are the *parts* of M. A compartite matrix with more than two parts is similarly defined. Suppose now that the factor-sequences of a latent root α with respect to the matrices Δ', Δ'' and M are respectively

$$R' = (r_0' \; r_1' \; r_2' \; \ldots)$$
$$R'' = (r_0'' \; r_1'' \; r_2'' \; \ldots)$$
$$R = (r_0 \; r_1 \; r_2 \; \ldots).$$

To obtain the sequence R from R', R'', we notice by inspection that the minor of any zero term in the determinant $|M - \lambda|$ vanishes. Hence it follows that any non-vanishing kth minor of $|M - \lambda|$, is a compartite determinant, the two parts of which are respectively, a μth minor of $|\Delta' - \lambda|$, and a $(k - \mu)$th minor of $|\Delta'' - \lambda|$ $(\mu = 0, 1, 2 \ldots k)$. Since $(\lambda - \alpha)^{r_k}$ is the highest power of $(\lambda - \alpha)$ which divides all the kth minors of $|M - \lambda|$, it follows from this that r_k is equal to the least of the numbers $(r'_\mu + r''_{k-\mu})$ $(\mu = 0, 1 \ldots k)$; that is to say, the sequence R is the *product* of the sequences R' and R''.

Thus we have :

> *The factor-sequence (with respect to a particular latent root) of a compartite matrix is the product of the factor-sequences of its parts.*

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* The term 'compartite matrix' (due to Dr. Cullis *Matrices and Determinoids*, Vol. 2) suggests that the matrix is built up all in *compartments*. The usual name for such matrices is 'the direct product'—which I believe is one of the worst specimens of bad naming in mathematics. The same name occurs in other branches—as the *direct product* of two linear algebras, and the *direct product* of two groups. In both cases, I think it might be advantageously replaced by 'compartite.'

The art of right naming appears to depend on a happy intuition of the essential feature of the object to be named. Sylvester is probably one of the best *namers* in mathematics. His names besides being always appropriate, are redolent of a lively picturesqueness. One could not but wish there were more of such names in mathematics.

§ 8. The Geometrical Theory of the Matrix.*

Let the matrix $M = |m_{rs}|$ of order n, be taken to represent the collineation

$$x'_k = m_{k1}\,x_1 + m_{k2}\,x_2 \cdots + m_{kn}\,x_n \quad (k = 1, 2 \ldots n)$$

in a space S_n of order n (or $n-1$ dimensions). Then to search for the points which are transformed into themselves—(the 'latent' points of the collineation), we put $x'_k = \lambda x_k$, and eliminating the x's we obtain the *characteristic* equation for λ :

$$|M - \lambda| = 0.$$

If the roots of this equation, that is, the *latent* roots of the matrix be $\lambda_1\ \lambda_2 \cdots \lambda_n$, then to each latent root λ_k corresponds in generally a definite latent point P_k, and *vice versa*. Suppose however that all† the n latent roots become equal to α. Then all the latent points P would coalesce and the actual manner of their coalescence could be investigated by examining what happens if all the latent roots are *nearly* equal to α. It would then appear that

(1) A certain number u_0 of the points P, say $P_1P_2 \cdots P_{u_0}$ lie in a *definite* region Q_{u_0} of order u_0, in the limit. Every point of Q_{u_0} is a latent point of the matrix, and Q_{u_0} itself is called the 'latent' region.

(2) The latent region Q_{u_0} together with a certain number t_1 of the remaining points P lies in a *definite* region Q_{u_1} of order $u_1 = u_0 + t_1$, in the limit. Qu_1 is called the semilatent region of the first species.

(3) Qu_1 together with a certain number t_2 of the remaining points P lies in a definite region Q_{u_2} (the semi-latent region of the second species) of order $u_2 = u_1 + t_2$, in the limit; and so on.

The sequence $U = (u_0\, u_1 \ldots)$, which we may call *the sequence of latent orders*, completely characterises the behaviour of the matrix in regard to its multiple latent root. This sequence being an increasing one, we use instead of it the *latency-sequence* T defined by

$$T = (t_0\, t_1\, t_2 \ldots);\ t_0 = u_0,\ t_1 = u_1 - u_0\ ;\ t_k = u_k - u_{k-1}.$$

From the property of the matrix, it may be shewn that the latency sequence T is a decreasing one. (*Cf.* Whitehead, *l.c.*)

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* For an exposition of this theory, see Whitehead's *Universal Algebra.*

† There is no loss of generality in this supposition, for a matrix with several distinct latent roots can be expressed as a compartite matrix, each part of which has only one latent root.

Further by reducing the matrix to a canonical form (as in Hilton : *Linear Substitutions*), it can be shewn that *the latency sequence corresponding to any latent root is the conjugate sequence of the invariant sequence for the same latent root.* This relation between the two sequences is a fundamental one.

§ 9. Geometrical Theory of Compartite Matrices.

Compartite matrices admit of a simple and natural interpretation in the geometrical theory. Let P_r, P_{n-r} be two non-intersecting regions of orders r and $n-r$ in a space S_n of order n, so that S_n is the *join* of P_r and P_{n-r}. Through any point of S_n there can be drawn one and only one straight line to meet P_r and P_{n-r}; hence any two collineations Δ', Δ'' in the spaces P , P_{n-r} respectively determine a unique collineation M of S_n. If the n points of reference for homogeneous co-ordinates in S_n are so chosen that r of them are points of reference in P_r and the remaining $n-r$ are the points of reference in P_{n-r}, then it will be immediately evident that the *matrix* M is the compartite matrix whose parts are the *matrices* Δ' and Δ''. Conversely, we may speak of the collineation Δ' as the P_r – component of the collineation M.

From this geometrical relation between the collineations M, Δ' and Δ'', it is immediately inferred that the semi-latent region of the kth species* of M is the join of the semi-latent regions of the kth species of Δ' and Δ''. Hence, the sequence of latent orders of a compartite matrix (with reference to a particular latent root) is the sum of the sequences of latent orders of its parts; and therefore it follows that *the latency sequence of a compartite matrix is the sum of the latency sequences of its parts.*

§ 10. Relations between the various sequences.

Suppose first that the algebraic view-point is fundamental. Then, from what has been proved (§ 7), the factor-sequence of a compartite matrix is the *product* of the factor-sequences of its parts. But the

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* If $l<k$ and the matrix Δ' has only l species of semi-latent regions, then for the validity of this theorem, the semi-latent region of the kth species of Δ' should be regarded as identical with that of the highest or the lth species.

factor-sequence is the integral of the invariant sequence. Hence by Theorem II, we have :

*The invariant sequence of a compartite matrix is the compound of the invariant sequences of its parts** ... ... (A)

Assuming now that the latency sequence is the conjugate of the invariant-sequence—we have, on using Theorem I, the theorem proved in § 9, namely,

The latency-sequence of a compartite matrix is the sum of the latency sequences of its parts... ... ... (B).

If on the other hand, the geometrical theory is considered fundamental, we could deduce (using the fact that the latency and invariant sequences are conjugates of one another), by using Theorems I and II, the theorem (A) and the theorem about the factor-sequence from (B).

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* This theorem is so fundamental in matrix-theory that it ought to find a place in all text-books on the subject. Hilton in his *Linear Substitutions* comes very near stating it, but changes his mind apparently at the last moment and passes on, leaving the reader in doubt. For a discussion of the theorem, see page 458 of the *Encyclopedie* already quoted.

Twin Points.

By M. Bhimasena Rao.

1. *Introduction.*

*Twin points** with respect to a triangle ABC, are diametrically opposite points on a rectangular hyperbola circumscribing the triangle. Since H, the orthocentre of the triangle, lies on every circumscribing rectangular hyperbola, points which are twins with respect to the triangle ABC are also twins with respect to the triangles HBC, HCA, HAB. Since the triangles formed by the points A, B, C, H, have a common nine-point centre, twins which are collinear with the nine-point centre have a symmetrical relation with respect to the four triangles of this orthocentric system.† In this paper are considered certain interesting geometrical properties of such twins and their isogonal conjugates with respect to the four triangles of the orthocentric system.

2. Theorem I.

If twins P, Q, with respect to a triangle ABC are collinear with N, the nine-point centre, and if P_1 is the isogonal conjugate of P, then PP_1 is parallel to HQ.

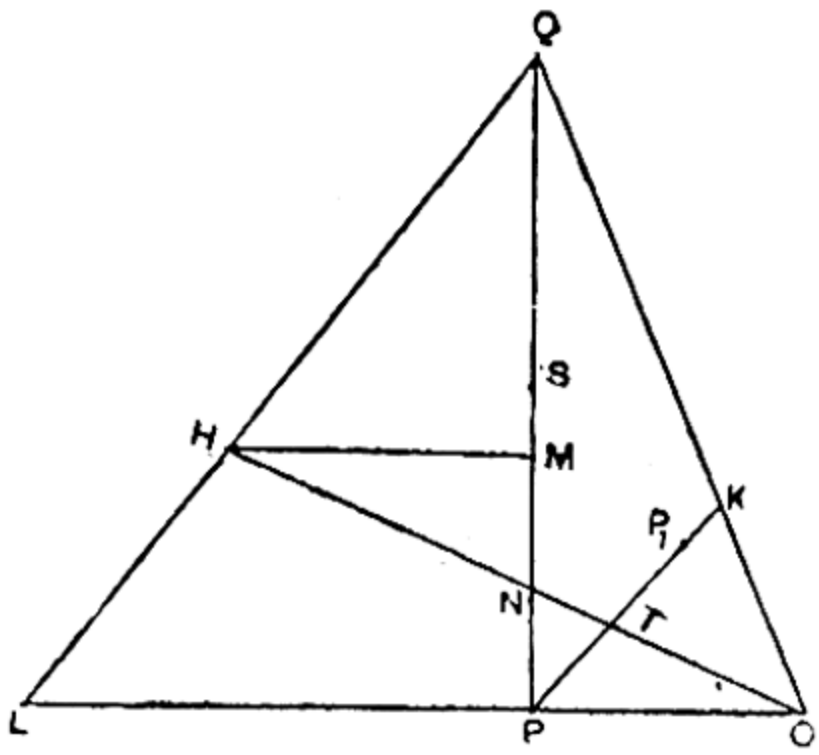


Fig 1.

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* W. Gallatly in his *Modern Geometry of the Triangle*, (second edition), pp. 62—64, defines twins as isogonal conjugates of a pair of inverse points, and mentions the interesting relation that twins have similar anti-pedal triangles, the difference of whose areas is four times the area of the triangle of reference.

† Such twins presented themselves in the study of *Morley's Theorem* published in the *Journal of the Indian Mathematical Society* for April 1920.

Let O be the circum-centre of ABC. Join OP and produce it to meet HQ in L; on PQ take NM equal to PN, and join HM. Let PP_1 meet OQ in K, and let S, T be the middle points of PQ, PP_1 respectively.

Since S, the centre of the rectangular hyperbola ABCPQ, lies on the nine-point circle,

$$R = 2NS = 2(NM + MS)$$
$$= PM + 2MS$$
$$= PS + MS$$
$$= QS + MS$$
$$= QM. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Since P and P_1 are isogonal conjugate points, we have by a theorem of Mr. V. Ramaswamy Aiyar,*

$$OP.OP_1 = 2R.NT$$
$$= 2QM.NT, \text{ by (1)}. \quad \ldots \quad \ldots \quad (2)$$

The pedal circle of P with respect to ABC passes through S. The radical axis of the pedal circle and the nine-point circle is perpendicular to NT, the join of their centres, and cuts the nine-point circle at an angle equal to SNT or its supplement. Hence by a result given by the author† angle POP_1 = angle SNT or its supplement. ... ... (3)

$$\therefore \quad \Delta(POP_1) = \tfrac{1}{2} OP.OP_1.\sin POP_1$$
$$= QM.NT.\sin SNT, \text{ by (2) and (3)}$$
$$= (QN - NM)\,NT.\sin SNT$$
$$= 2(QNT) - 2(MNT)$$
$$= 2(QMT).$$

Since T is the middle point of PP_1,

$$\Delta(POP_1) = 2\Delta(POT)$$
$$\therefore \quad \Delta(POT) = \Delta(QMT) \quad \ldots \quad \ldots \quad (4)$$

$$\text{Again } \frac{OK}{KQ} = \frac{\Delta(POT)}{\Delta(PTQ)} = \frac{\Delta(QMT)}{\Delta(PTQ)} = \frac{QM}{QP}, \text{ by (4)} \quad \ldots \quad (5)$$

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* *Vide* W. Gallatly : *Modern Geometry of the Triangle*, p. 79, 2nd Edition.

† *Vide J. I. M. S.*, Question 876.

Since ON = NH, and PN = NM, HM is equal and parallel to OP.

$$\therefore \quad \frac{QM}{QP} = \frac{HM}{PL} = \frac{OP}{PL}.$$

$$\therefore \quad \frac{OK}{KQ} = \frac{QM}{QP} = \frac{OP}{PL}, \text{ by (5).}$$

Hence PK is parallel to QL. That is, PP_1 is parallel to QH.

3. THEOREM II.

If Q_1 is the isogonal conjugate of Q in Theorem I, then PP_1 and QQ_1 intersect on the circum-circle of ABC.

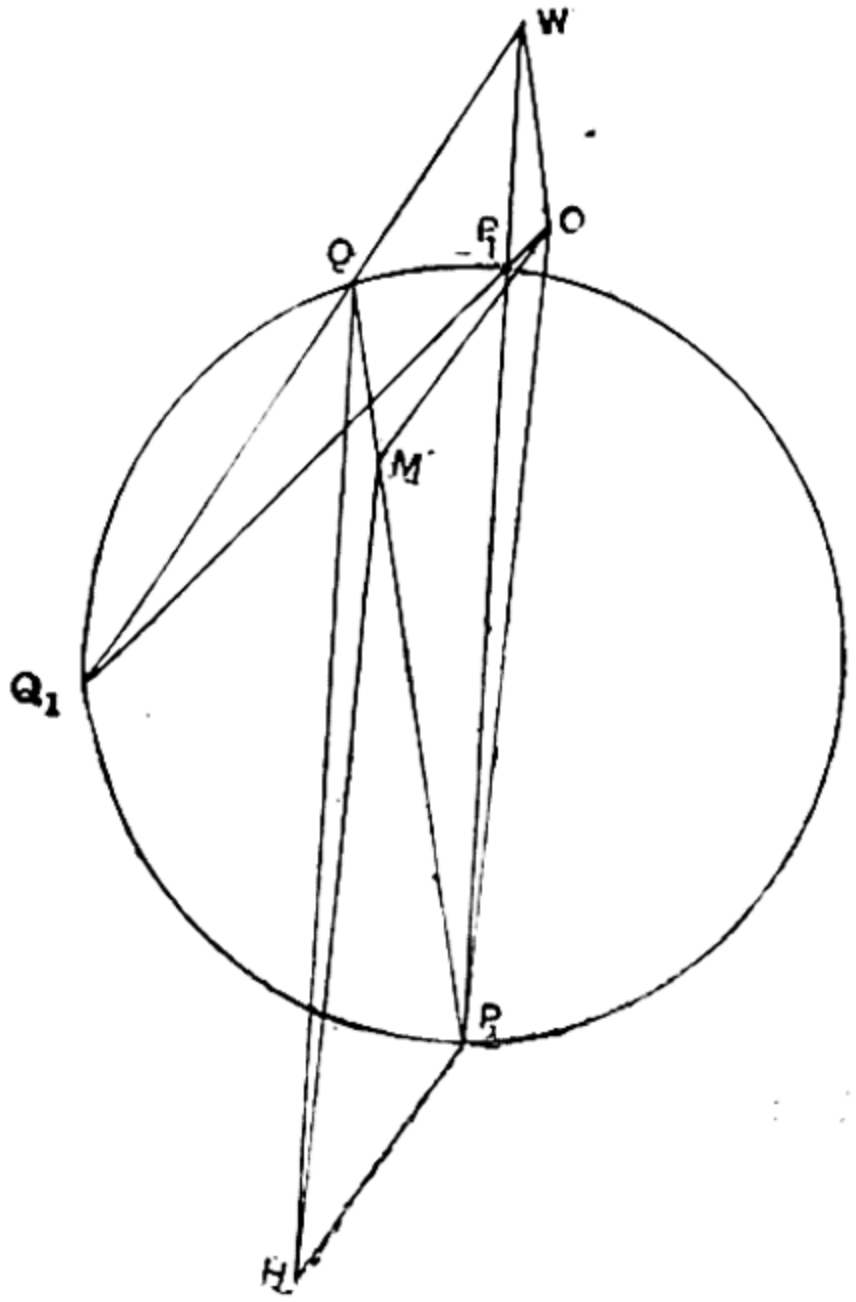


Fig. 2.

Let W be the point of intersection of PP_1 and QQ_1.

By Theorem I,

HP is parallel to QQ_1,

and HQ is parallel to PP_1

$\therefore$ HQWP is a parallelogram.

Since PM and OH bisect each other at N,

HMOP is a parallelogram.

$\therefore$ MQWO is a parallelogram. ... ... (6)

Hence OW is equal and parallel to QM which has been proved in (1) to be equal to the radius of the circle ABC. Therefore the joins of isogonal pairs (PP_1) and (QQ_1) intersect on the circumcircle of ABC.

4. THEOREM III.

Twins collinear with the nine-point centre are concyclic with their isogonal conjugates.

With the previous notation, since P_1, Q_1 are isogonal conjugates of twin points, they are inverse points with respect to the circle ABC.*

$\therefore$ $OP_1 . OQ_1 = R^2 = OW^2$, by Theorem II

$\therefore$ angle OWP = angle OQ_1W.

Since OW is parallel to PQ, by (6),

angle OWP = angle QPP_1

$\therefore$ angle P_1Q_1Q = angle QPP_1.

Hence P, Q, P_1, Q_1 are concyclic, which proves the theorem.

The converse results corresponding to Theorems II and III are true, but cannot be proved by retracing the steps. We shall give a separate demonstration using an interesting theorem in the theory of similar figures.

5. THEOREM IV.

If twins are concyclic with their isogonal conjugates, then the joins of isogonal pairs intersect on the circum-circle of the triangle of reference.

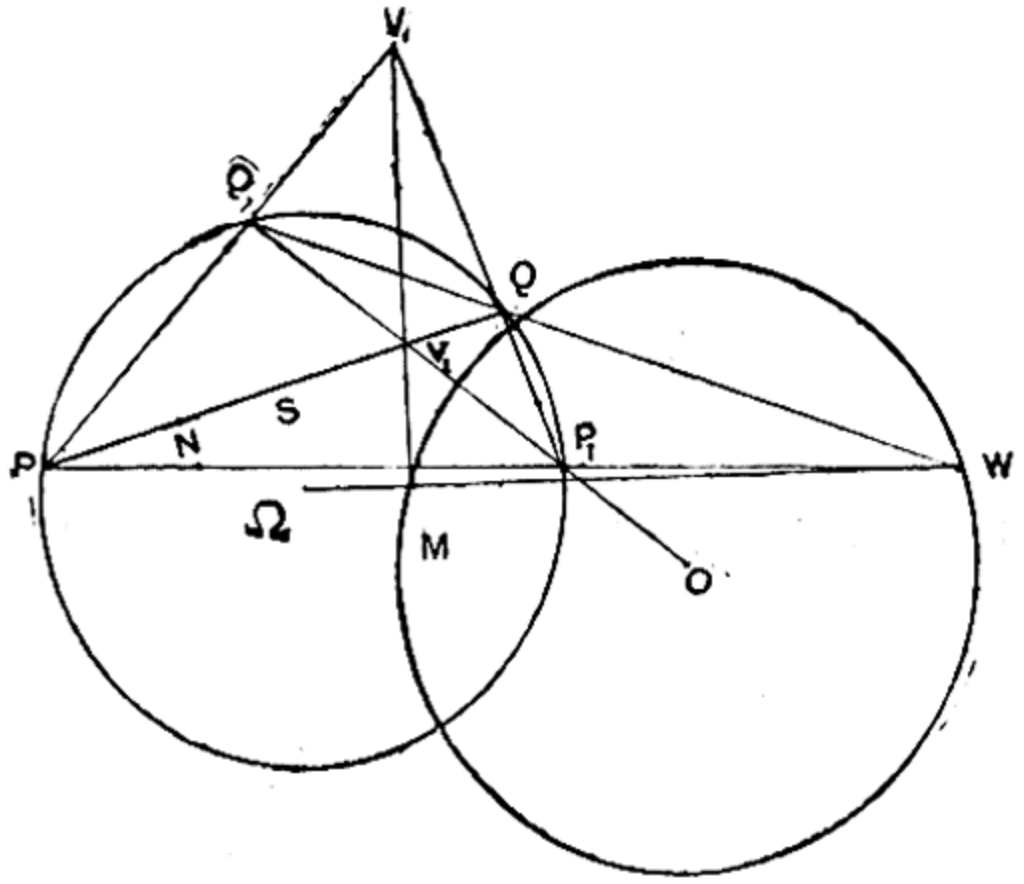


Fig. 3.

* *Vide* Gallatly : *Modern Geometry of the Triangle*, p. 62,

In the figure, P, Q are the twins; P_1, Q_1 their isogonal conjugates, with respect to a triangle ABC, and W the point of intersection of the joins of isogonal pairs PP_1, QQ_1; V and V_1 are the meets of PQ_1, QP_1, and PQ, P_1Q_1 respectively. The centre of the circle PQ_1QP_1 is Ω, and ΩW cuts VV_1 at M.

Now, VV_1W is the diagonal triangle of PQ_1QP_1.

$\therefore$ VV_1 is the polar of W with respect to the circle (Ω), and M is the inverse of W with respect to the same circle.

Next, consider directly similar figures described on the chords PQ and Q_1P_1. Their double point is the inverse, with respect to the circle (Ω), of the meet of the joins of the non-corresponding pairs PP_1 and QQ_1.*

Hence M is the double point. ... ... (7)

Again, P,P_1 and Q,Q_1 are isogonal pairs with respect to the triangle ABC. The double point of directly similar figures described on PQ and Q_1P_1 lies, by a theorem of Mr. V. Ramaswamy Aiyar, on the circum-circle of ABC.† Therefore the double point M lies on the circle ABC. (8)

Since P_1, Q_1 are the isogonal conjugates of P, Q, they are inverse points with respect to the circle ABC. The circle PQ_1QP_1 passing through these inverse points is orthogonal to the circle ABC. Since M lies on the circle ABC by (8), W which is the inverse point of M with respect to the circle PQ_1QP_1 lies on the circle ABC, and thus the theorem is proved.

6. THEOREM V.

If twins are concyclic with their isogonal conjugates, then their join passes through the nine-point centre.

Referring to figure 3, it is easily seen that the points V,V_1 are also isogonal conjugates with respect to the triangle ABC.‡ The rectangular hyperbola ABCPQ is the isogonal transformation of the line OP_1Q_1 with respect to the triangle ABC. Since V_1 lies on the line OP_1Q_1, the rectangular hyperbola passes through V. ... ... (9)

Again, in the circle PQ_1QP_1,

angle P_1PQ_1 + angle P_1QQ_1 = 2 right angles;

i.e. angle VPW + angle VQW = 2 right angles.

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* *Vide. J.I.M.S.*, Vol. IV. Page 175.

† *Vide J.I.M.S.*, Vol. II, Question 125.

‡ *Vide J.I.M.S.*, Vol. IV, Page 173,

Hence V being a point on the rectangular hyperbola ABCPQ by (9), and the line VW subtending supplementary angles at the diametrically opposite points P and Q of this hyperbola, it follows that the hyperbola passes also through W. ... ... (10)

Since P_1, Q_1 are inverse points with respect to the circle ABC,

$$OP_1 . OQ_1 = R^2 = OW^2, \text{ by Theorem IV.}$$

$$\therefore \text{ angle } OWP_1 = \text{angle } OQ_1W = \text{angle } QPW,$$

in the cyclic quadrilateral QQ_1PP_1.

Hence OW is parallel to PQ. ... ... (11)

Now, the fourth point of intersection of the rectangular hyperbola ABCPQ and the circle ABC is W. If S is the centre of the rectangular hyperbola, we know that the point H on WS produced such that HS is equal to SW is the orthocentre of the triangle ABC.

In the triangle OWH, S is the middle point of WH, and the line QSP, being parallel to OW by (11), bisects OH. Hence PQ passes through the nine-point centre.

7. THEOREM VI.

If P_1, P_2, P_3, P_4 and Q_1, Q_2, Q_3, Q_4 are respectively the isogonal conjugates of the twins P, Q, (collinear with the nine-point centre), with respect to the four triangles of the ortho-centric system, then the points P, Q, P_1, P_2, P_3, P_4 and P, Q, Q_1, Q_2, Q_3, Q_4 lie on two rectangular hyperbolas concentric with the rectangular hyperbola PQABCH.

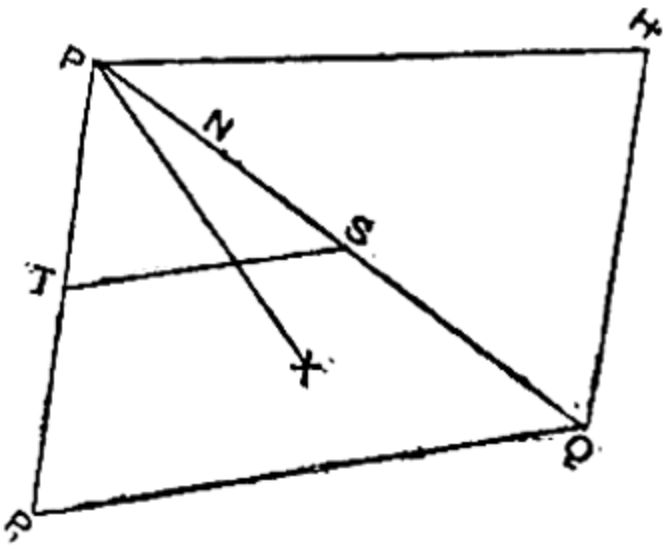


Fig. 4.

Draw the tangent PX to the rectangular hyperbola ABCPQ. Let P_1 be the isogonal conjugate of P with respect to the triangle ABC, and S, T the middle points of PQ, PP_1 respectively.

Since PQ is a diameter of the rectangular hyperbola and H is a point on it,

$$\text{angle } HPX = \text{angle } HQP.$$

Because HQ and PP_1 are parallel, by Theorem I,

$$\text{angle HQP} = \text{angle } P_1PQ.$$

$$\therefore \quad \text{angle HPX} = \text{angle } P_1PQ. \quad \ldots \quad (12)$$

The pedal circle of P cuts the nine-point circle at S. The angle of intersection of the two circles is, by a theorem of Mr. V. Ramaswamy Aiyar,* equal to

$$90° - (\text{PAB} + \text{PBC} + \text{PCA}).$$

Hence $$\text{NST} = 90° - (\text{PAB} + \text{PBC} + \text{PCA}). \quad \ldots \quad (13)$$

If the direction angles of the tangent PX, referred to the triangle ABC be α, β, γ then

$$\text{angle PAB} = \text{angle APX} - \alpha$$
$$\text{,, PBC} = \text{,, BPX} - \beta$$
$$\text{,, PCA} = \text{,, CPX} - \gamma.$$

$$\therefore \quad \text{NST} = 90 + \alpha + \beta + \gamma - (\text{APX} + \text{BPX} + \text{CPX}).$$

Since TS is parallel to P_1Q,

$$\text{angle NST} = \text{angle } P_1QP.$$

From (12) and (13), we have

$$P_1PQ - P_1QP = HPX + APX + BPX + CPX - (90° + \alpha + \beta + \gamma). \quad (14)$$

Next, consider the reciprocal of the nine-point circle with respect to the rectangular hyperbola ABCPQ. Since the hyperbola is self-conjugate with respect to the ortho-centric triangle, the reciprocal is a parabola, having S, the centre of the hyperbola, for its focus. The polar of the nine-point centre N is the directrix, and is parallel to the tangent PX, since S, N, P are in a line. If the direction angles of PX with respect to the ortho-centric triangle be α', β', γ', then

$$\text{angle SPX} = 90° - (\alpha' + \beta' + \gamma').\dagger \quad \ldots \quad (15)$$

Since the sides of the orthocentric triangle cut the corresponding sides of the triangle ABC at angles B − C, C − A, A − B, we have,

$$90 - (\alpha' + \beta' + \gamma') = 90 - (\alpha + \beta + \gamma).$$

Hence $$\text{QPX} = \text{SPX} = 90 - (\alpha + \beta + \gamma). \quad \ldots \quad (16)$$

Substituting in (14), we get

$$P_1PQ - P_1QP = HPX + APX + BPX + CPX + QPX - 180°. \quad (17)$$

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* *Vide J.I.M.S.*, Vol. IX, Page 167.

† *Vide* Gallatly's *Modern Geometry of the Triangle*, § 41, Page 27.

The symmetry of the result (17) shows that the difference of the base angles of the triangles P_1PQ, P_2PQ, P_3PQ, P_4PQ, is the same. By a well known result, the points P_1, P_2, P_3, P_4 lie on a rectangular hyperbola having PQ as a diameter. Similarly Q_1, Q_2, Q_3, Q_4, lie on another rectangular hyperbola having PQ as a diameter.

8. THEOREM VII.

If the pedal circles of P with respect to the four triangles of the orthocentric system are drawn, then the inverses of P with respect to these circles are collinear with the centre of the rectangular hyperbola ABCPQ.

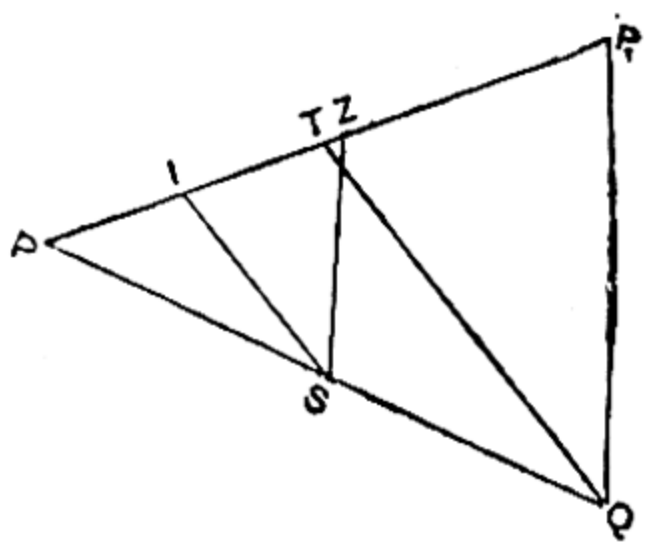


Fig. 5.

Let T be the centre of the pedal circle of P with respect to the triangle ABC, and I the inverse of P with respect to the pedal circle. Draw the tangent QZ to the rectangular hyperbola $PQ\,P_1P_2P_3P_4$. Join S, the centre of the rectangular hyperbola to T and I.

Since TS is the radius of the pedal circle of P,

$$TI.\ TP = TS^2.$$

$$\therefore \quad \text{angle } TSI = \text{angle } TPQ.$$

Since QZ is a tangent to the rectangular hyperbola PQP_1,

$$\text{angle } P_1QZ = \text{angle } P_1PQ.$$

$$\therefore \quad \text{angle } TSI = \text{angle } P_1QZ$$

But TS is parallel to P_1Q;

$$\therefore \quad SI \text{ is parallel to } QZ.$$

Hence I, the inverse of P, lies on the diameter conjugate to PQ in the rectangular hyperbola $P_1P_2P_3P_4$, and the result stated in the theorem easily follows.

When P is an arbitrary point, it may be shown that the inverses of P with respect to the pedal circles of P drawn as before, will lie on a circle passing through S, the centre of the rectangular hyperbola ABCP.

The present theorem shows that when P, S, N are collinear, the circle degenerates into a line.

9. Let us now reciprocate the result stated in the last theorem, with respect to a circle, centre P. The points A, B, C, H reciprocate into four lines, having P as a common point of intersection of the director circles of the conics touching these lines. The rectangular hyperbola ABCP reciprocates into a parabola touching these lines and having as focus the inverse of S,—in other words, the inverse of S is the Miquel point of the four lines. The line of collinearity of the four inverses of P referred to in the last theorem, inverts into a circle passing through P and the Miquel point. This circle is the centre circle of the quadrilateral formed by the lines. The nine-point circle reciprocates into an inconic of the diagonal triangle of the quadrilateral, having P as focus and the Miquel point as vertex in virtue of the collinearity of P, S, N. Hence we obtain the following result :—

THEOREM VIII.

P is a common point of intersection of the director circles of the system of conics touching the sides of a quadrilateral whose diagonal triangle is XYZ. If the conic inscribed in XYZ with P as focus has its vertex at the Miquel point of the quadrilateral, then the centre circle of the quadrilateral passes through P.

10. Next, consider the centre circle of the quadrilateral, referred to in Theorem VIII. It is the locus of the double foci of all circular cubics passing through the six points of intersection of the sides of the quadrilateral, because tangents at two of the common points of a pencil of cubics intersect on a conic passing through these two points. The ninth common point of the above pencil of circular cubics is the Miquel point. Under the conditions of Theorem VIII the point P lies on the centre circle. Hence with P as double focus, a circular cubic can be described passing through the six points of intersection of the sides of the quadrilateral and the Miquel point. It has been shown that the normal to the cubic at the Miquel point passes through the double focus P*. Since the inconic of the diagonal triangle having P as focus has its vertex at the Miquel point, the circular cubic touches the inconic and its auxillary circle at the Miquel point.

Reciprocating the cubic with respect to a circle centre P, it is easily seen that we obtain Morley's theorem, *viz.*

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* *Vide* : The author's paper on *Morley's Theorem*, *J. I. M. S.*, April 1920.

Every curve of class three which touches the six joins of four orthocentric points and passes through the circular points touches the nine-point circle.

P being the double focus of the circular cubic is also the double focus of the curve in Morley's theorem. The above discussion shows that the point of contact of the nine-point circle and the curve lies on the line joining the nine-point centre and the double focus of the curve, and that it is the centre of the rectangular hyperbola passing through the orthocentric points and the double focus. The inverse of a circular cubic with respect to its double focus being a bi-circular quartic having the same double focus, the inverse theorem may be stated thus*:—

THEOREM IX.

If twin points P, Q, are collinear with the nine-point centre, a bi-circular quartic can be drawn with P as double focus passing through P and the feet of the perpendiculars from P on the six joins of the orthocentric points A, B, C, H; this quartic passes through the centre of the rectangular hyperbola ABCP, and touches the nine-point circle at that point.

We have also established incidentally

THEOREM X.

Twins collinear with the nine-point centre are the double foci of the curves in Morley's theorem.

11. We shall now conclude with some remarks regarding the equation to the locus of the double foci referred to in the last theorem.

Taking the orthocentric triangle DEF as a triangle of reference, the trilinear equation to the locus is easily seen to be

$$\begin{vmatrix} \cos D & \cos E & \cos F \\ x & y & z \\ \dfrac{\sin D}{y^2-z^2} & \dfrac{\sin E}{z^2-x^2} & \dfrac{\sin F}{x^2-y^2} \end{vmatrix} = 0;$$

i.e., $$\frac{\sin D.(y\cos F - z\cos E)}{y^2-z^2} + \text{two similar terms} = 0. \quad \ldots \quad (18)$$

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* If instead of the double focus, the centre of inversion is a single focus, the corresponding theorem has been stated by Mr. V. Ramaswami Aiyar, as follows:—

Given a triangle ABC, if a point P and the feet of the perpendiculars from it on the sides and altitudes of ABC all lie on a cartesian oval, then the negative pedal of the oval with respect to P touches the nine-point circle. (*Vide J.I.M.S.*, Volume 10, Question 960)

This is a curve* of the fifth degree circumscribing the orthocentric triangle and having double points at A, B, C, H $(1, \pm 1 \pm 1)$. It passes through the nine-point centre (cos D, cos E, cos F). It is obvious geometrically that the nine-point centre and the sixteen in- and ex-centres of the four triangles of the orthocentric system lie on the curve, as in these cases the joins of the twins are easily seen to pass through the nine-point centre,—a fact noticed also by Mr. H. W. Richmond, F. R. S †

The curve is circular, and its remaining three points of intersection with the line of infinity lie on M'Cay's cubic ‡ of the orthocentric triangle. It touches the nine-point circle at the circular points and the remaining six points of intersection of the curve and the nine-point circle lie on another interesting cubic which is the locus of the centres of conics touching the sides of the orthocentric triangle at points where the normals are concurrent.§ In fact, equation (18) can be transformed into

$$C.S = I.Q. \quad \ldots \quad \ldots \quad (19)$$

where $C \equiv \cos \mathrm{D}\, x(y^2 - z^2) + \ldots\ldots\ldots + \ldots\ldots\ldots = 0$ is the M'Cay's cubic of DEF,

$S \equiv yz \sin \mathrm{D} + zx \sin \mathrm{E} + xy \sin \mathrm{F} = 0$ is the nine-point circle of ABC,

$I \equiv x \sin \mathrm{D} + y \sin \mathrm{E} + z \sin \mathrm{F} = 0$ is the line at infinity,

and $Q \equiv \cos \mathrm{D}\, yz(y^2 - z^2) + \ldots\ldots\ldots + \ldots\ldots\ldots = 0,$

is a circular quartic, which may be written as

$$\frac{S.R - I.M}{\lambda},$$

where $R \equiv x^2(\sin^2 \mathrm{E} - \sin^2 \mathrm{F}) + \ldots\ldots\ldots + \ldots\ldots\ldots = 0$ is the rectangular hyperbola ABCN,

$M \equiv \sin \mathrm{E} . \sin \mathrm{F} . x(y^2 - z^2) + \ldots\ldots + \ldots\ldots\ldots = 0$ is the second cubic referred to above,

and λ is the constant $2 \sin \mathrm{D} . \sin \mathrm{E} . \sin \mathrm{F}$.

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* The line at infinity forms part of the locus of the double foci inasmuch as any point on it may be regarded as the double focus of the three-cusped hypocycloid belonging to the system, and with the curve given in (18) constitutes the complete locus.

† *Vide* Edinburgh Mathematical Society's *Proceedings* for 1919—20, Vol. XXXVIII, pp. 2—5.

‡ The locus of isogonal conjugate points whose pedal circle touches the nine-point circle, *Vide* Gallatly's *Modern Geometry of the Triangle*, Second Edition, p. 80,

§ *Vide J. I. M. S.*, Question 430, wherein some properties of this cubic are stated.

Hence the equation to the curve may be written in the form,

$$\lambda\, CS - ISR + I^2M = 0. \qquad \ldots \qquad \ldots \quad (20)$$

An interesting form of the equation is obtained by the use of covariants of two conics S, S′ and the line $lx + my + nz$.

This form is

$$S^2L' + S'^2L = 2SS' \frac{L\Sigma' + L'\Sigma}{\varphi} \qquad \ldots \quad (21)$$

where S,S' are any two rectangular hyperbolas through A, B, C, H, in point co-ordinates, and Σ, Σ', their expressions in line co-ordinates,

$lx + my + nz$ is the line at infinity,

J is the Jacobian of S,S' and $lx + my + nz$,

L is the line joining the poles of $lx+my+nz$ with respect to J and S,

L' is the corresponding line with respect to S′,

and φ is the harmonic envelope of S and S'.*

If we take S and S' as the rectangular hyperbolas touching the line at infinity, then Σ and Σ' will vanish, L and L' will become the isotropic lines through the nine-point centre, and the equation (21) simply reduces to

$$S^2L' + S'^2L = 0, \qquad \ldots \qquad \ldots \quad (22)$$

putting in evidence the fact that the nine-point centre lies on the curve and that it is the double focus of the curve.

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* *Vide* Salmon's *Conic Sections*, (1904), pages 344 and 363.

The result (21) is easily verified by taking

$$S = z^2 - x^2,\ S' = x^2 - y^2,$$
$$l = \sin D,\ m = \sin E,\ n = \sin F.$$

Whence
$$\Sigma = -\sin^2 E,\ \Sigma' = -\sin^2 F,$$
$$\phi = -\sin^2 D + \sin^2 E + \sin^2 F,$$

and
$$L = \sin E\,(z\cos D - x\cos F)$$
$$L' = \sin F\,(x\cos E - y\cos D).$$

Definition of Straight Line *

By S. V. Ramamurthy, M.A., I.C.S.

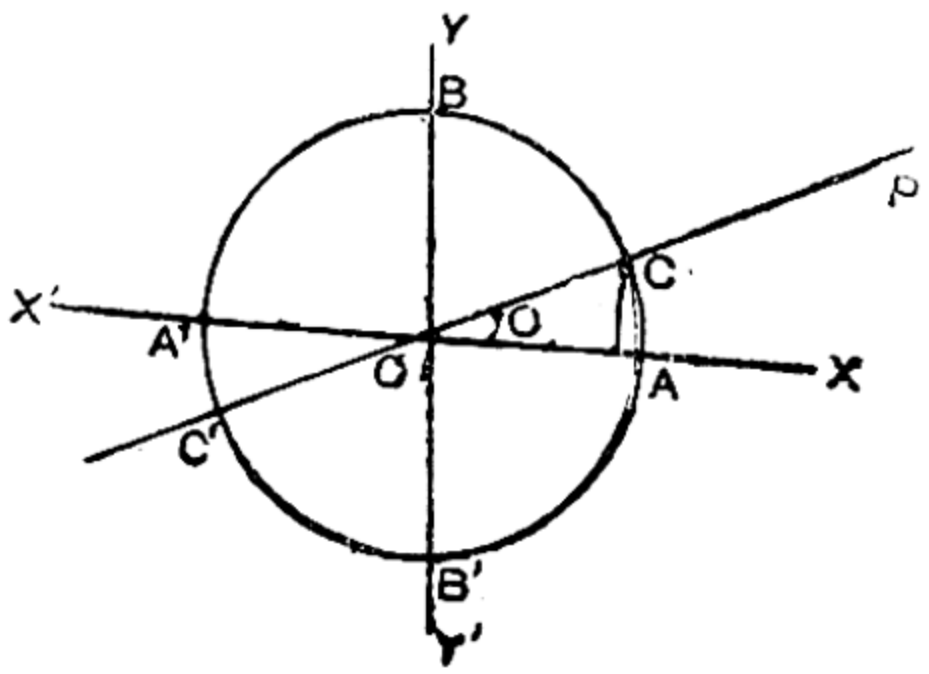
1. In his excellent little book "*Euclid's Parallel Postulate*" (The Open Court Publishing Company, 1905), Withers has stated that "the essential meaning of the straight line as used in the various forms of geometry has not been clearly determined." Lobatchewsky defines the straight line "as one which fits upon itself in all its positions so that, if we turn the surface containing it about two points of the line, the line does not move." This may be as correct a definition as that of man as a featherless biped; but each gives equally little information about the essential nature of straight line or man. Euclid defines a straight line as the shortest distance between two points. But to define distance we need a straight line. Some are content to say that a straight line has the same direction throughout its length. But then direction is left undefined. We also say that a straight line has two directions.

There is thus no satisfactory definition of a straight line at present.

To analyse the nature of a straight line, take a straight line XOX′ and another perpendicular to it, YOY′, as the axes of x, y.

Describe a circle of unit radius cutting these axes at A,A′; B,B′. Let a line OP cut the circle at C, C′.

Draw the co-ordinates of C.



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* Presented to the Fourth Conference of the Indian Mathematical Society, Poona, April 1924.

Let $\angle XOP = \theta$, $OA = 1$, $OB = i$, $OC = x + iy = e^{i\theta}$, $OA' = -1$, $OB' = -i$, $OC' = -e^{i\theta}$.

Along OX, mark off unit lengths AA_1, A_1A_2 ..... So too along OY unit lengths BB_1, B_1B_2 ...; and along OP unit lengths CC_1, C_1C_2, ...

Let OA now represent an atom of time. Then OX is a line of time particles and the lines OA, OA_1, OA_2, ... can be represented by the natural numbers 1, 2, 3 .....

So too, the lines OB, OB_1, OB_2, can be represented by i, $2i$, $3i$,..... If here, we take i itself as the unit, the lines are represented by the natural numbers 1, 2, 3, ...

So too, the lines OC, OC_1, OC_2 ..... can be represented by $e^{i\theta}$, $2e^{i\theta}$, $3e^{i\theta}$ ..... or taking $e^{i\theta}$ as unit, by the natural numbers 1, 2, 3 .....

Now, in my paper on the 'Definition of Number,' I have defined a natural number as a piece of time. Hence any part of the line OX is a piece of time. Hence OX is a line of time. So too OY and OP are lines of time.

No curved line through O can have its parts so represented by natural numbers. For, if we mark off unit lengths on it, $e^{i\theta}$ (which may be called the *density* of the length in relation to the plane) alters at each part of the curved line.

Thus we can define a straight line as a line of time.

This definition applies to any kind of continuum — of time, space, matter or mind. For, in all such continua there are time particles. A line of time particles gives a straight line.

In a space-time continuum of two dimensions, the space particles are all equal and a line of them has its parts represented by natural numbers and yields what may be regarded as a line of time. Hence, we have two straight lines determined by the space particles and the time particles. The theory of dimensions based on the presence of independent sets of equal particles has already been described in my paper on "Time, Space, Matter and Mind." The view of straight line presented here is consistent with that theory of dimensions and supports it.

SRINAGAR.

2. Next, let us consider how a straight line has *two directions.*

In the above figure, let OA = r. Then OA′ = $-r$. If t be a particle of time and r be the corresponding particle of space, then

$$t = kr^2,$$

where k is a constant. (See *Journal of the Indian Mathematical Society*, Vol. XV, No. 1, page 10).

Hence t of OA $= kr^2$
$= k(-r)^2$
$= t$ of OA′.

That is, the time value corresponding to two equal lengths along OX, OX′ is the same.

Hence if we draw the lines of time corresponding to the straight lines OX, OX′, we get the same line of time.

Hence the straight lines which can be drawn from O in a Euclidean space continuum of a three dimensional world are arranged in pairs each yielding one line in the equivalent time continuum. Such straight lines we define as parts of the same straight line of different directions.

So also in a Euclidean three dimensional continuum of matter in a four-dimensional continuum, straight lines through a point O are arranged in trios, each yielding one line in the equivalent time continuum.

For, $$M^3 = (\wedge M)^3 = (\curlywedge M)^3$$

and each gives the same t. ($\wedge$, $\curlywedge$ are dinus and trinus signs. See *J.I.M.S.*, Vol. XV, p. 28).

Thus in Time, Space and Matter continua, Euclidean under the guidance of a perpendicular direction, there are straight lines radiating from a given point which are all of different directions in time, which are arranged in pairs of same direction in space and which are arranged in trios of the same direction in matter.

(i) A straight line in time has one direction : positive.

(ii) A straight line in space has two directions : positive and negative.

(iii) A straight line in matter has three directions : positive, dinitive and trinitive.

A Classical Indian Puzzle-Problem.

By A. A. KRISHNASWAMI AYYANGAR, M.A., L.T.,

1. In the *Messenger of Mathematics* Vol. LIII., No. 2, Dr. Glaisher has discussed some of the puzzle-problems of Widmont and Blasius. This reminds us of an interesting variant of those problems occurring in some of the classical Indian treatises on Arithmetic, such as *Bijaganita* of Bhâskara and *Ganitasara-sangraha* of Mahâviràchàrya. Both these works are about a thousand years old; but the problem we shall refer to is perhaps of a much more ancient origin; for, we find Bhaskara (about 1050 A. D.) citing it as one instanced by earlier authors.

The problem is this:

Three traders, with six, eight and hundred *panas* as capital, respectively, having bought fruits at a uniform rate, sold a part in lots and disposed of the remainder at one for five *panas* and thus became *equally* rich. What were the purchase and sale prices ?

The rule for solving such problems is enunciated by Vishnugupta, Mahaviracharya and others, thus :

The required sale-price is any number greater than the greatest of the given capitals, while the purchase-price is less than the product of the two sale-prices by unity.

Bhaskara, however, attempts the solution of a set of linear indeterminate equations using the Indian *kuttaka* or continued fraction method and finally arrives at the correct result, though in the course of his reasoning he makes serious mistakes.

The solution (given by Vishnugupta and others) is the same as that given by Dr. Glaisher's '*quotient*' system (so called, because the figures in the second column, *vide infra*, are the quotients obtained by dividing those in the first column by p) on p. 19, Vol. LIII, No. 2, *Messenger of Mathematics*:

$p\mu - 1$	$\mu - 1,\ p : p - 1;\ \mu$	$p\mu - 1$
$2(p\mu - 1)$	$2\mu - 1,\ p : p - 2;\ \mu$	,,
.....	.....	.....
$p(p\mu - 1)$	$p\mu - 1,\ p : \quad 0 : \mu$	,,

In the above tables μ corresponds to the Indian residual (retail) sale-price, p articles per *pana* is the sale-price in lots, and $(p\mu - 1)$ articles per *pana* the purchase-rate, while the co-efficients of $(p\mu - 1)$ in the first column represent the capitals and the numbers in the last column the common realised amount.

The General Problem.

2. We shall now discuss the general problem of n capitals ($c_1, c_2, \ldots\ldots c_n$, in ascending order of magnitude) and the residual sale-price μ and give all the classes of solutions in terms of the '*quotient*', as well as the '*non-quotient*' systems.

Denote the purchase-price by x, the sale-price in lots by p, the number of articles sold in lots by the several tradesmen by $pa_1, pa_2, \ldots\ldots pa_n$, and the corresponding residual articles by $b_1, b_2, \ldots\ldots b_n$; and the problem reduces itself to the solution, in positive integers, of the following $(2n - 1)$ indeterminate equations :—

$$\left.\begin{aligned} c_1x = pa_1 + b_1\,;\ c_2x = pa_2 + b_2\,;\ \ldots\ldots\ c_nx = pa_n + b_n\,;\\ a_1 + \mu b_1 = a_2 + \mu b_2 = \ldots\ldots = a_n + \mu b_n \end{aligned}\right\} \quad \text{(A)}$$

where $c_1 < c_2 < c_3 \ldots\ldots < c_n.$

From the equations (A), we derive

$$\left.\begin{aligned} (b_1 - b_2)(p\mu - 1) &= (c_2 - c_1)\,x\\ (b_2 - b_3)(p\mu - 1) &= (c_3 - c_2)\,x\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ (b_{n-1} - b_n)(p\mu - 1) &= (c_n - c_{n-1})\,x \end{aligned}\right\} \quad \ldots \quad \ldots \quad \text{(B)}$$

By addition,

$$(b_1 - b_n)(p\mu - 1) = (c_n - c_1)\,x$$

Let h be the H. C. F. of $c_2 - c_1, c_3 - c_2, \ldots\ldots c_n - c_{n-1}$

Then hx will be divisible by $p\mu - 1$.

Set $$hx = k(p\mu - 1).$$

Then $$b_1 - b_n = \frac{(c_n - c_1)\,k}{h}.$$

$$\therefore \quad b_1 \geqslant \frac{(c_n - c_1)\,k}{h}. \quad \ldots \quad \ldots \quad (1)$$

Since $$c_1 x = pa_1 + b_1,$$

$$c_1 x \geqslant b_1\,;$$

i. e., $$\frac{c_1 k(p\mu - 1)}{h} \geqslant \frac{(c_n - c_1)\,k}{h}.$$

$$\therefore \quad p \geqslant \frac{c_n}{c_1\mu}. \quad \ldots \quad \ldots \quad (2)$$

Now, by giving values 1, 2, 3, ... for k and for p any integral value not less than $\frac{c_n}{c_1\mu}$, so that

$$k(p\mu - 1) \equiv 0 \pmod{h},$$

the corresponding values of x may be determined and hence those of a_1 and b_1 from the equations

$$pa_1 + b_1 = c_1 x = \frac{k(p\mu - 1)c_1}{h}.$$

Obviously, a_1 can take all values 0, 1, 2, ... up to

$$I\left\{\frac{1}{p}\left[\frac{k(p\mu - 1)c_1}{h} - \frac{(c_n - c_1)k}{h}\right]\right\}$$

i. e.,

$$I\left\{\frac{k}{ph}(p\mu c_2 - c_n)\right\},$$

where $I(m)$ represents the integral part of m.

The values of $(a_1 + \mu b_1)$ vary with those of a_1 and b_1.

Thus, when $a_1 = r$,

$$b_1 = \frac{k(p\mu - 1)c_1}{h} - pr,$$

and

$$a_1 + \mu b_1 = (p\mu - 1)\left(\frac{\mu k c_1}{h} - r\right).$$

The least value of $(a_1 + \mu b_1)$ occurs when a_1 is a maximum,

i. e., when

$$a_1 = I\left\{\frac{k}{ph}(p\mu c_1 - c_n)\right\}.$$

When a_1, b_1, are determined, equations (B) give the values of b_2, b_3, ... from which the corresponding values of a_2, a_3, ... are readily gathered.

Thus the equations (A) are completely solved.

Note: If d be the highest common factor of μ and h, k also must be a multiple of d and so k cannot take all values 1, 2, 3, ...

3. *Classification of Solutions*:

(i) If

$$\frac{(c_n - c_1)k}{h} \geqslant p \geqslant \frac{c_n}{c_1\mu},$$

the corresponding solutions belong entirely to the non-quotient system, since $b_1 \geqslant p$; and for a given set of values k and p (satisfying, of course, the limitations of § 2) there are

$$1 + I\left\{\frac{k}{ph}(p\mu c_1 - c_n)\right\}$$

non-quotient systems satisfying the equations (A).

(ii) If p be greater than both $\frac{c_n}{c_1\mu}$ and $\frac{(c_n - c_1)k}{h}$, there may be, for a given set of values of k and p, one quotient system and

$$I\left\{\frac{k\,(p\mu - 1)\,c_1}{ph}\right\}$$

non-quotient systems, or merely

$$I\left\{\frac{k(p\mu - 1)c_1}{ph}\right\}$$

non-quotient systems satisfying (A).

(iii) If $k\mu c_1 \leqslant h$, it is easily seen that

$$I\left\{\frac{k}{ph}(p\mu c_1 - c_n)\right\}$$

vanishes and therefore there is *only one* solution in the quotient system and *none* in the non-quotient system.

4. *Illustration*:

In the problem cited by Bhaskara, $c_1 = 6$, $c_2 = 8$, $c_3 = 100$, $\mu = 5$.

The H. C. F. of $(c_2 - c_1)$ and $(c_3 - c_1)$ is 2, *i.e.*, $h = 2$;

and $$p \geqslant \frac{100}{6 \times 5} \text{ or } \geqslant 4;$$

and $$b_1 \geqslant \frac{(100 - 6)\,k}{2} \text{ or } 47\,k.$$

(i) When $47\,k \geqslant p \geqslant 4$, the solution is exclusively in the non-quotient systems; thus let $k = 1$, then since $x = \frac{5p - 1}{2} =$ an integer, $v \equiv 1 \pmod 2$ and $p = 5$ is the least value admissible; there are

$$1 + I\left\{\frac{1}{5 \times 2}(5 \times 5 \times 6 - 100)\right\}$$

or 6 non-quotient systems which are solutions of the problem (*vide* § 3 (i)). These are exhibited below in a tabular form:—

[C = capital; P = purchase-price; F = number of fruits bought;
L = number sold in lots; R = residual articles;
A = amount realised.]

Non-Quotient Systems.

Serial No.	C	P = x	F = CP	L	R	A = 5R + $\frac{L}{p}$
1	6	12	72	0	72	360
	8	12	96	25	71	360
	100	12	1200	1175	25	360
2	6	12	72	5	67	336
	8	12	96	30	66	336
	100	12	1200	1180	20	336
3	6	12	72	10	62	312
	8	12	96	35	61	312
	100	12	1200	1185	15	312
4	6	12	72	15	57	288
	8	12	26	40	56	288
	100	12	1200	1190	10	288
5	6	12	72	20	52	264
	8	12	96	45	51	264
	100	12	1200	1195	5	264
6	6	12	72	25	47	240
	8	12	96	50	46	240
	100	12	1200	1200	0	240

(ii) When $p \geqslant 4$ and also $> 47\,k$, there may or may not be a solution in the quotient system. For, let $k = 1$ and $p = 49$; then

$$x = \frac{49 \times 5 - 1}{2} = 122.$$

Now proceeding to solve the equations

$$49\,a_1 + b_1 = 6x = 732,$$

we find that for a solution in the quotient system, b_1 is equal to the least positive remainder obtained by dividing 732 by 49.

$$\therefore \qquad b_1 = 46,$$

which does not satisfy the condition

$$b_1 \geqslant 47\,k,$$

since $k = 1$.

Thus there is no solution in the quotient systems for $k = 1$, $p = 49$.

In this case also, therefore, the solutions are all in the non-quotient systems of which there are

$$I\left(\frac{49\times 5-1}{2\times 49}\times 6\right)$$

or 14 according to the formula of § 3. (ii).

These correspond to the values 0, 1, 2, 3,.....13 for a_1 and the realised amounts

3660, 3416, 3172, 2928, 2684, 2440,
2196, 1952, 1708, 1464, 1220, 976, 732, 488 and 244.

If however $k = 1$, $p = 51$, then $x = 127$ and the remainder obtained by dividing 762 by 51 is 48, which is greater than 47, and hence there is one solution in the quotient-system, *viz.*

C	P = x	F = CP	L	R	A = 5R + $\frac{L}{p}$.
6	127	762	714	48	254
8	127	1016	969	47	254
100	127	12700	12699	1	254

and $I\left(\frac{762}{51}\right)$ *i.e.*, 14 solutions in the non-quotient system, corresponding to the realised amounts

508, 762, 1016, 1270, 1524, 1778, 2032,
2286, 2540, 2794, 3048, 3302, 3556 and 3810.

5. For the existence of a solution in the quotient-system, it is necessary and sufficient, that the remainder obtained by dividing

$$\frac{k(p\mu-1)c_1}{h} \text{ by } p$$

should be greater than or equal to

$$\frac{(c_n-c_1)k}{h}.$$

It is not possible to obtain an expression for the form of this remainder except in the following two cases :—

Case I. If $\frac{kc_1}{h} < p$ and is an integer, the remainder is easily seen to

be $p - \frac{kc_1}{h}$ and hence the necessary and sufficient condition for a solution in the quotient-system is in this case

$$p - \frac{kc_1}{h} \geqslant \frac{(c_n - c_1)k}{h};$$

i.e.

$$p \geqslant \frac{c_n k}{h},$$

in addition to the condition that $\frac{kc_1}{h}$ is an integer.*

These conditions are evidently satisfied by taking

$$k = h \text{ and } p \geqslant c_n.$$

Then

$$x = p\mu - 1.$$

This is just the Indian solution, where $p > c_n$ in order to avoid the zero-remainder. Thus, in Bhaskara's problem, if we take

$$k = 2, p = 100, \text{ then } x = 499;$$

and the solutions represented below involve a zero-remainder :—

C	P	F	L	R	$A = 5R + \frac{L}{p}$.
6	499	2994	2900	94	499
8	499	3992	3900	92	499
100	499	49900	49900	0	499

while if

$$k = 2, p = 101, \text{ then } x = 504$$

and the solutions represented below have no such zero-remainder.

C	P	F	L	R	$A = 5R + \frac{L}{p}$.
6	504	3024	2929	95	504
8	504	4032	3939	93	504
100	504	50400	50399	1	504

This is, in fact, the solution given by Bhaskara.

* The two conditions $p \geqslant \frac{c_n k}{h}$ and $\frac{kc_1}{h}$ = an integer imply the other condition $p \geqslant \frac{c_n}{c_1 \mu}$, for $kc_1 > h$ and therefore $kc_1\mu > h$, that is, $c_n \frac{k}{h} > \frac{c_n}{c_1 \mu}$.

Case II. If $\frac{kc_1}{h} < p$ but not an integer, let h_1 be the remainder when $k\mu c_1$ is divided by h; then the remainder when $\frac{k(p\mu - 1)c_1}{h}$ an integer is divided by p is either

$$\frac{ph_1 - kc_1}{h}, \text{ when } ph_1 > kc_1;$$

or
$$p + \frac{ph_1 - kc_1}{h}, \text{ when } ph_1 < kc_1.$$

The following sets of conditions are necessary and sufficient for a quotient system in this case :—

(1) $$p \geqslant \frac{kc_n}{h_1}; \quad p \geqslant \frac{c_n}{c_1\mu}$$

which is included in the former when $k\mu c_1 \geqslant h$ (*vide* § 2).

or (2) $$p \geqslant \frac{kc_n}{h+h_1}; \quad p < \frac{kc_1}{h_1}; \quad p \geqslant \frac{c_n}{c_1\mu}; \quad p > \frac{kc_1}{h}.$$

The latter conditions imply the further relation

$$\frac{kc_n}{h+h_1} < \frac{kc_1}{h}, \text{ i.e., } c_n < c_1\left(1 + \frac{h}{h_1}\right).$$

Cor.: (i) p cannot take any value between

$$\frac{kc_1}{h_1} \text{ and } \frac{kc_n}{h_1}.$$

(ii) If $\frac{c}{c_1} < 1 + \frac{h_1}{h}$ and $k\mu c_1 > h$, it follows easily that $\frac{kc_1}{h}$ is greater than both $\frac{kc_n}{h+h_1}$ and $\frac{c_n}{c_1\mu}$. Thus, in this case the total number of values of p between $\frac{kc_1}{h}$ and $\frac{kc_1}{h_1}$ giving the quotient systems are

$$I\left\{\frac{kc_1(h - h_1)}{h^2h_1}\right\}, \text{ or } 1 + I\left\{\frac{kc_1(h - h_1)}{h^2h_1}\right\}.$$

Illustration.

Let $$c_1 = 100, \; c_2 = 113, \; c_3 = 139, \; \mu = 20.$$

Then $h = 13$; giving k the value 10, we notice that p should satisfy the equation

$$20p \equiv 1 \text{ (mod. 13)},$$

i.e.
$$p \equiv 2 \text{ (mod. 13)}.$$

Now $\frac{kc_1}{h} = \frac{1000}{13} = 76\frac{12}{13}$; $h_1 = 6$; $\frac{kc_1}{h_1} = 166\frac{2}{3}$;

$$\frac{kc_n}{h+h_1} = 73\frac{3}{19}; \quad \frac{c_n}{c_1\mu} = \frac{139}{2000}; \quad \frac{kc_n}{h_1} = 231\frac{2}{3}.$$

The values of p greater than $\frac{kc_1}{h_1}$ admissible in this problem are the integers greater than 231 which are congruent to 2 (mod. 13), *viz.*, the infinite series in A. P with common difference 13, beginning with 236.

For example, when $p = 236$, $x = 3630$ and the solution is as follows :

C	P	F	L	R	A
100	3630	363000	362968	32	2178
113	„	410190	410168	22	„
139	„	504570	504568	2	„

The values of p greater than $\frac{kc_1}{h}$ and less than $\frac{kc_1}{h_1}$ are given by the integers between $76\frac{12}{13}$ and $166\frac{2}{3}$ which are congruent to 2 (mod. 13) *viz.*, the seven integers

80, 93, 106, 119, 132, 145, 158.

NOTE 1. It is obvious that when $\frac{kc_1}{h} = p$ or a multiple of p, the least value of $b_1 = 0$ and there is no solution in the quotient system.

NOTE 2. When $\frac{kc_1}{h} > p$, it is not possible to define the limits of p in terms of the constants k, h, h_1, μ, c_1, c_2, ... and therefore the existence of a solution in the quotient system has to be verified only by trial. Thus for the problem discussed above, the values of

$$p < \frac{kc_1}{h} \text{ or } 76\frac{12}{13}$$

which are likely to give a solution in the quotient-system are the integers congruent to 2 (mod. 13) and lying between

$$\frac{(c_3 - c_1)k}{h} \text{ and } \frac{kc_1}{h}$$

i.e., between 30 and $76\frac{12}{13}$, *viz.*, 41, 54 and 67.

But it can be verified that the remainder obtained by dividing $\frac{k(p\mu - 1)c_1}{h}$, *i.e.*, $\frac{1000(20p - 1)}{13}$ by p is not greater than 30 for any of the above values of p.

Thus the only solutions in the quotient-system corresponding to the value 10 of k are those in which $p = 80, 93, \ldots\ldots$ up to 158 and the terms in the infinite A.P with common difference 13, beginning with 236; 80 is therefore the least value of p when $k = 10$.

Summable Double Series.*

By V. TIRUVENKATACHARYA, M.A., L.T.

1. Professors Bromwich and Hardy have shown in the *Proceedings* of the London Mathematical Society, Series 2, Vol. 2, that, by extending the theorems of Frobenius and Holder to double series, it is possible to find the sum of the oscillatory series

$$\begin{aligned}
&1 - 1 + 1 - 1 + 1 - \ldots\ldots \\
&- 1 + 1 - 1 + 1 - 1 + \ldots\ldots \\
&+ 1 - 1 + 1 - 1 + 1 - \ldots\ldots \\
&- 1 + 1 - 1 + 1 - 1 + \ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

and have obtained the sum of this series to be $\frac{1}{4}$. In this paper, it is attempted to develop a general theory by which we can find the sum of this and other oscillatory double series. The discussion in the following pages closely follows the treatment in §§ 99—103 of Bromwich in his *Theory of Infinite Series.*

2. GENERAL THEORY :—*Double Integral for Summable Double Series.* [The double integral being of the same type as Borel's integral for single summable series, the integral may be called Borel's double integral.]

Following Prof. Hardy in his definition of single summable series, let us enunciate the following principle† :— If two limiting processes, performed in a definite order on a function lead to a definite value A, but, when performed in a reverse order lead to a meaningless result B, then we may agree to interpret B as meaning A.

For example, the equation

$$\lim_{(m,n)\to\infty}\left[\int_0^x\int_0^\infty f(x,y,m,n)\,dxdy\right] = \int_0^x\int_0^\infty \left[\lim_{(m,n)\to\infty} f(x,y,m,n)\right] dxdy \quad 2.1$$

cannot be true always. But it may happen that the expression on the right is perfectly definite, while that on the left is not. We may agree, then, according to the principle enunciated above, to take the right-hand integral as defining the left-hand limit.

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* Read before the Fourth Conference of the Indian Mathematical Society, Poona, April 1924.

† *Vide* Bromwich : *Theory of Infinite Series*, p. 267.

[Here $f(x, y, m, n)$ is a function of four variables and the two limiting processes, are (a) making m and n tend to ∞ and (b) the double integration with respect to x and y. The conditions which $f(x,y,m,n)$ must satisfy in order that the double integral may be unique, irrespective of the order of integration, and the double limit unique, irrespective of the order of making m and n tend to infinity, will be specified in the course of the paper.]

Suppose that $f(x, y, m, n)$ is the sum of $(m + 1)$ rows and $(n + 1)$ columns of a double series of terms which are functions of x and y.

$$\begin{aligned} \text{Let } f(x, y, m, n) = {} & \phi_{00}(x, y) + \phi_{01}(x, y) + \quad \dots \quad + \phi_{0n}(x, y) \\ & + \phi_{10}(x, y) + \quad \dots \quad \dots \quad + \phi_{1n}(x, y) \\ & + \quad \dots \quad \dots \quad \dots \\ & + \phi_{m0}(x, y) + \quad \dots \quad \dots \quad + \phi_{mn}(x, y) \end{aligned}$$

and

$$\mathrm{F}(x, y) = \lim_{(m, n) \to \infty} f(x, y, m, n)$$

where $f(x, y, m, n)$ is such that the repeated and the double limits exist and are equal.

These conditions are sufficient to give a unique value for it, *viz.* $\mathrm{F}(x, y)$.

Now equation 2.1 becomes

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[\int_0^{\infty}\int_0^{\infty} \phi_{mn}(x, y)\, dxdy\right] = \int_0^{\infty}\int_0^{\infty} \mathrm{F}(x, y)\, dxdy. \qquad 2.2$$

In this equation, the double integral on the right may happen to be convergent, whereas the double series on the left may not be so. In that case, we agree to take the integral as defining the sum of the oscillatory series on the left.

[Here, conditions on $\mathrm{F}(x, y)$ must be imposed in order that the order of integration may be changed, without altering its value. The conditions referred to are that $\int_0^{\infty} \mathrm{F}(xy)\, dx$, $\int_0^{\infty} \mathrm{F}(xy)\, dy$ and either

$$\int_0^{\infty} dx \int_0^{\infty} \mathrm{F}(xy)\, dy, \text{ or } \int_0^{\infty} dy \int_0^{\infty} \mathrm{F}(xy)\, dx$$

must be convergent.]

The special type of ϕ's that will be taken now is

$$\phi_{m,n}(x, y) = e^{-x-y} \frac{u_{m,n} x^m y^n}{m!\, n!}$$

where u_{mn} is independent of both x and y.

Now we see that

$$\int_0^\infty\int_0^\infty \phi_{m,n}\,dxdy = \int_0^\infty\int_0^\infty e^{-x-y}\,\frac{u_{m,n}x^m y^n}{m!\ n!}\,dxdy = u_{m,n} \quad . \qquad 2.3$$

$$\text{and } F(x, y) = e^{-x-y}\begin{vmatrix} u_{00} + u_{01}y + u_{02}\dfrac{y^2}{2!} + \ldots\ldots\ldots + u_{0n}\dfrac{y^n}{n!} + \ldots\ldots \\ + u_{10}x + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + u_{m0}\dfrac{x^m}{m!} + \ldots\ldots\ldots\ldots\ldots\ldots + \dfrac{u_{mn}x^m y^n}{m!\ n!} \ldots\ldots \end{vmatrix}$$

$$= e^{-x-y}\,u(x, y). \ldots \quad \ldots \quad \ldots \quad \ldots \qquad 2.4$$

It is to be assumed that the co-efficients $u_{m,n}$ are such that the double series is convergent for all values of x and y in Pringsheim's sense; that is, the rows and the columns converge and the double series is convergent.

Then, provided that $\int_0^\infty \int_0^\infty e^{-x-y}\,u(x, y)\,dxdy$ is convergent, we may agree to associate its value with the double series $\sum_0^\infty \sum_0^\infty u_{m,n}$ if this is non-convergent, and the integral may be called the *sum* of the series. Let the sum be denoted by the symbol $\overset{\infty}{S}\,\overset{\infty}{S}\,(u_{m,n})$.

3. CONDITION OF CONSISTENCY: Our definition will be an absurdity unless $S\,S\,(u_{m,n})$ agrees with $\Sigma\Sigma\,u_{m,n}$ whenever the latter converges to a *definite* sum, that is, if $\Sigma\Sigma\,u_{mn}$ converges to a definite sum s then

$$\int_0^\infty \int_0^\infty e^{-x-y}\,u(x, y)dxdy$$

must have the same value s.

Since $u_{m,n}$ is convergent and satisfies Pringsheim's conditions, we can find numbers μ and ν such that

$$|\,u_{m,n}\,| < \varepsilon, \text{ if } m > \mu \text{ and } n > \nu.$$

$$\text{Then} \quad \left|\sum_m^\infty \sum_n^\infty u_{mn}e^{-x-y}\,\frac{x^m y^n}{m!\ n!}\right| < \varepsilon \left| e^{-x-y}\sum_m^\infty \sum_n^\infty \frac{x^m y^n}{m!\ n!}\right|$$

$$< \varepsilon, \text{ if } m > \mu \text{ and } n > \nu. \qquad 3.1$$

Therefore the double series converges uniformly for all values of x and y within the intervals (o, p), (o, q), where p and q may be arbitrarily large.

$$\therefore \int_0^p \int_0^q e^{-x-y}\, u(x,y)\,dx\,dy = \sum_0^\infty \sum_0^\infty u_{m,n} \int_0^p \int_0^q e^{-x-y}\, \frac{x^m y^n}{m!\ n!}\, dx\,dy.^* \qquad 3.2$$

$$= \sum_0^\infty \sum_0^\infty u_{mn}\, p_m q_n \qquad \ldots \qquad \ldots \qquad 3.3$$

where $$p_m q_n = \int_0^p \int_0^q \frac{e^{-x-y}\, x^m y^n}{m!\ n!}\, dx\,dy$$

$$= e^{-p-q} \left(\frac{p^{m+1}}{(m+1)!} + \frac{p^{m+2}}{(m+2)!} \cdots \right) \left(\frac{q^{n+1}}{(n+1)!} + \ldots \right) \qquad 3.4$$

It is evident that $p_m q_n$ is a decreasing sequence and can be arranged as a double series

$$e^{-p-q} \left| \begin{array}{l} \dfrac{p^{m+1} q^{n+1}}{(m+1)!\,(n+1)!} + \left(\dfrac{p^{m+1} q^{n+2}}{(m+1)!\,(n+2)!} + \ldots\ldots\ldots \right. \\ + \dfrac{p^{m+2} q^{n+1}}{(m+2)!\,(n+1)!} + \dfrac{p^{m+2} q^{n+2}}{(m+2)!\,(n+2)!} + \ldots\ldots\ldots \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array} \right| \quad \ldots \quad 3.5$$

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* Since $$\Sigma\Sigma\, u_{mn}\, e^{-x-y}\, \frac{x^m y^n}{m!\ n!}$$

is uniformly convergent, μ and ν can be chosen such that

$$\sum_m^\infty \sum_n^\infty u_{m,n}\ e^{-x-y}\, \frac{x^m y^n}{m!\ n!} < \varepsilon, \text{ whenever } m > \mu \text{ and } n > \nu.$$

Hence we have

$$\left| \sum_m^\infty \sum_n^\infty \int_0^p \int_0^q u_{mn}\, e^{-x-y}\, \frac{x^m y^n}{m!\ n!}\, dx\,dy \right| < \varepsilon\, pq.$$

At the same time we have

$$\left| e^{-x-y}\, u(x,y) - \sum_0^{m-1} \sum_0^{n-1} u_{m,n}\, e^{-x-y}\, \frac{x^m y^n}{m!\ n!} \right| < \varepsilon$$

$$\therefore \left| \int_0^p \int_0^q e^{-x-y}\, u(xy) - \int_0^p \int_0^q \sum_0^{m-1} \sum_0^{n-1} u_{mn}\, \frac{e^{-x-y} x^m y^n}{m!\ n!} \right| < \varepsilon\, pq.$$

i.e., $$\left| \int_0^p \int_0^q e^{-x-y}\, u(x,y) - \sum_0^{m-1} \sum_0^{n-1} u_{m,n} \int_0^p \int_0^q \frac{e^{-x-y} x^m y^n}{m!\ n!} \right| < \varepsilon\, pq.$$

$$\therefore \left| \int_0^p \int_0^q e^{-x-y}\, u(x,y) - \sum_0^\infty \sum_0^\infty u_{mn} \int_0^p \int_0^q \frac{e^{-x-y} x^m y^n}{m!\ n!} \right| < 2\varepsilon\, pq.$$

Since p and q are positive and the double sequence $p_m q_n$ is never increasing such that

$$p_v q_\mu - p_{v+1} q_\mu, \; p_v q_\mu - p_v q_{\mu+1},$$

and $$p_v q_\mu - p_v q_{\mu+1} - p_{v+1} q_\mu + p_{v+1} q_{\mu+1},$$

are positive, applying the extended form of Abel's lemma for double series, we have

$$h(p_0 q_0 - p_m q_n) + h_{mn} p_m q_n < \sum_0^\infty \sum_0^\infty u_{mn} p_m q_n < H(p_0 q_0 - p_m q_n) + H_{mn} p_m q_n. \quad \ldots \quad 3.6$$

where h and H are the lower and the upper limits of $\sum_0^x \sum_0^y u_{xy}$ as x varies from 0 to $m-1$ and y varies from 0 to $n-1$; h_{mn} and H_{mn} are the lower and upper limits of the same as x varies from m to ∞ and y from n to ∞.

Now if $\Sigma\Sigma u_{mn}$ converges to the sum s in Pringsheim's sense, we can find m and n corresponding to a given ε such that $h_{mn} \geqslant s - \varepsilon$ and $H_{mn} \leqslant s + \varepsilon$ however small ε be.

As p and q tend to ∞,

$$\lim. \; p_0 q_0 = 1 \text{ and } \lim. \; p_m q_n = 1.$$

For,

$$p_0 \, q_0 \; e^{p+q} = pq + \frac{pq^2}{1.2!} + \frac{pq^3}{1.3!} + \ldots\ldots$$
$$+ \frac{p^2 q}{2!\,1!} + \frac{p^2 q^2}{2!\,2!} + \ldots\ldots\ldots\ldots$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$= (e^p - 1)(e^q - 1).$$

$$\therefore \quad \lim_{p,\, q \to \infty}. \; p_0 \, q_0 = \lim_{p,\, q \to \infty}. \quad (e^p - 1)(e^q - 1)\, e^{-p-q} = 1.$$

Similarly

$$\lim. \; p_m \, q_n = \lim. \, e^{-p-q} \left(e^p - 1 - \frac{p}{1} - \frac{p^2}{2!} - \ldots - \frac{p^{m-1}}{m-1!}\right) \times$$
$$\left(e^q - 1 - \frac{q}{1} \ldots\ldots\ldots - \frac{q^{n-1}}{n-1!}\right) = 1]$$

Hence $$\lim_{p,\, q \to \infty}. \quad h\,(p_0 \, q_0 - p_m \, q_n) + h_{mn} \, p_m \, q_n \geqslant s - \varepsilon$$

and $$\lim_{p,\, q \to \infty}. \quad H\,(p_0 \, q_0 - p_m \, q_n) + H_{mn} \, p_m \, q_n \leqslant s + \varepsilon.$$

$$\therefore \quad s - \varepsilon \leqslant \lim_{p,\, q \to \infty} \Sigma\Sigma u_{mn}\, p_m\, q_n \leqslant s + \varepsilon.$$

But ε being arbitrary, the max. and min. limits are equal.

$$\therefore \quad \lim_{p,\, q \to \infty} \overset{\infty}{\underset{0}{\Sigma}}\overset{\infty}{\underset{0}{\Sigma}} u_{mn}\, p_m\, q_n = s.$$

But, if $\Sigma\Sigma u_{mn}$ diverges, say to $+\infty$, we can find m and n such that $h_{mn} > \mathrm{N}$, however great N may be.

$$\therefore \quad \lim_{p,\, q \to \infty} h\,(p_0\, q_0 - p_m\, q_n) + h_{mn}\, p_m\, q_n = h_{mn} > \mathrm{N}.$$

$$\therefore \quad \lim_{p,\, q \to \infty} \Sigma\Sigma u_{mn}\, p_m\, q_n > \mathrm{N}.$$

Since N is arbitrary, this implies that $\Sigma\Sigma u_{mn}\, p_m q_n$ tends to infinity with p and q.

4. Let us apply the theory developed above to find the sum of certain oscillatory double series.

(1) Let $S =$ $1 - 1 + 1 - 1 + \ldots\ldots$
$- 1 + 1 - 1 + 1 \ldots\ldots\ldots$
$+ 1 - 1 + 1 - 1 \ldots\ldots\ldots$
$- 1 + 1 - 1 + 1 \ldots\ldots\ldots$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Now $u\,(x, y) = 1 - \dfrac{x}{1!} + \dfrac{x^2}{2!} + \ldots\ldots\ldots\ldots$

$$- \frac{y}{1!} + \frac{xy}{1!\,1!} - \frac{x^2 y}{2!\,1!} \ldots\ldots$$

$$+ \frac{y^2}{2!} - \frac{xy^2}{1!\,2!} + \ldots\ldots\ldots\ldots$$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$$= e^{-x-y}.$$

$$\therefore \quad S = \int_0^\infty \int_0^\infty e^{-x-y}\, u\,(x, y)\, dx dy$$

$$= \int_0^\infty \int_0^\infty e^{-2x-2y}\, dx dy = \tfrac{1}{4}.$$

(2) Again, let $S = 1 - 1 + 1 - 1 + \ldots\ldots\ldots$
$- 2 + 2 - 2 + 2 - \ldots\ldots\ldots$
$+ 2^2 - 2^2 + 2^2 - 2^2 \ldots\ldots\ldots$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$$\text{Here } u(x,y) = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots$$
$$- \frac{2y}{1!} + \frac{2xy}{1!\,1!} - \frac{2x^2y}{2!\,1!} + \frac{2x^3y}{3!\,1!} - \ldots$$
$$+ \frac{2^2y^2}{2!} - \frac{2xy^2}{1!\,2!} + \frac{2x^2y^2}{2!\,2!} - \ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$= e^{-x-2y}.$$

$$\therefore \quad S = \int_0^\infty \int_0^\infty e^{-x-y}\, u(x,y)\, dxdy = \int_0^\infty \int_0^\infty e^{-2x-3y}\, dxdy = \tfrac{1}{2}.\tfrac{1}{3} = \tfrac{1}{6}.$$

(3) If $S = 1 - 2 + 3 - 4 + \ldots\ldots$

$$- 2 + 3 - 4 + 5 - \ldots\ldots$$
$$+ 3 - 4 + 5 - \ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

then $u(x,y) = e^{-x-y}(1 - x - y)$.

$$\therefore \quad S = \int_0^\infty \int_0^\infty e^{-2x-2y}(1 - x - y)\, dxdy = 0.$$

5. Summability by Rows and by Columns:

Let us consider the $(r+1)$th row of the double series $\sum_0^\infty \sum_0^\infty (u_{mn})$.

It is

$$u_{r,0} + u_{r,1} + u_{r,2} + u_{r,3} + \ldots\ldots$$

which can be represented by the symbol $\sum_{n=0}^{n=\infty} (u_{rn})$, provided the row is summable.* The sum of the double series can now be taken as

$$\sum_{r=0}^{\infty} \left[\sum_{n=0}^{\infty} u_{rn} \right]$$

provided

$$v_0 + v_1 + v_2 + \ldots\ldots\ldots\ldots$$

is summable, where

$$v_r = \sum_{n=0}^{\infty} (u_{rn}).$$

We get a similar result by considering the columns.

It can be seen that the result is

$$\sum_{n=0}^{\infty} \left[\sum_{r=0}^{\infty} u_{rn} \right]$$

provided that the columns and the sums of the columns are summable. Now the question arises whether

$$\sum_{r=0}^{\infty} \left[\sum_{n=0}^{\infty} u_{rn} \right] \text{ and } \sum_{n=0}^{\infty} \left[\sum_{r=0}^{\infty} u_{rn} \right]$$

---

* What is meant by summability here, is that, even though the series is oscillatory or divergent, the corresponding integral is convergent.

are equal to each other. It can be verified that this is true for the examples dealt with in § 4 of this paper.

To prove that this is true generally, let us take the row corresponding to $\overset{\infty}{S}_{n=0} (u_{rn})$ in F (x, y). It is

$$e^{-x-y}\frac{x^r}{r!}\left(u_{r,0} + u_{r,1}\frac{y}{1!} + u_{r,2}\frac{y^2}{2!} + \ldots\ldots\right) \qquad 5.1$$

$$= e^{-x-y}\frac{x^r}{r!}.\ \mathrm{U}_r(y), \text{ (say).}$$

Since we have assumed $\overset{\infty}{S}_{n=0} (u_{rn})$ to be summable,

$$\left(\int_0^\infty e^{-y}\,\mathrm{U}_r(y)\,dy\right)$$

is convergent.

The sum for the $(r+1)$th row is given by the integral

$$e^{-x}\frac{x^r}{r!}\int_0^\infty e^{-y}\,\mathrm{U}_r(y)dy.$$

Hence the sum of the double series by rows is

$$\int_0^\infty \left[e^{-x}\frac{x^r}{r!}\int_0^\infty e^{-y}\mathrm{U}_r(y)dy\right]dx$$

and its sum by columns is

$$\int_0^\infty \left[e^{-y}\frac{y^n}{n!}\int_0^\infty e^{-x}\mathrm{U}_n(x)dx\right]dy.$$

Since we have assumed u_{mn} to be such that the double series

$$\sum_0^\infty\sum_0^\infty u_{mn}x^m y^n$$

converges, and converges by rows as well as by columns, the double series converges to the same sum, whether taken by rows or by columns.

$$\therefore \qquad \Sigma\frac{x^r}{r!}\,\mathrm{U}_r(y) = \Sigma\frac{y^n}{n!}\,\mathrm{U}_n(x) = u(x, y).$$

Hence the two double integrals for the sum of the summable series are

$$\int_0^\infty e^{-x}\,dx\int_0^\infty e^{-y}\,u(x, y)dy \text{ and } \int_0^\infty e^{-y}\,dy\int_0^\infty e^{-x}\,u(x, y)dx$$

which can be written also as

$$\int_0^\infty dx\int_0^\infty e^{-x-y}\,u(x, y)dy \text{ and } \int_0^\infty dy\int_0^\infty e^{-x-y}\,u(x, y)dx.$$

Since the series is summable by rows

$$\int_0^\infty e^{-x-y}\, u(x, y)dy$$

must converge, and since it is summable by columns

$$\int_0^\infty e^{-x-y}\, u(x, y)dx$$

must converge, and since the double series itself is summable, the corresponding double integral

$$\int_0^\infty \int_0^\infty e^{-x-y}\, u\,(x, y)\, dxdy$$

is convergent. And de la Vallee Poussin has shown that, if the three integrals mentioned above are convergent, then

$$\int_0^\infty dx \int_0^\infty e^{-x-y}\, u\,(x, y)\, dy = \int_0^\infty dy \int_0^\infty e^{-x-y}\, u\,(x, y)\, dx.$$

Hence for an oscillatory summable double series summable* by rows and by columns, sum of the double series is equal to the sum obtained, either by rows or by columns.

In concluding, it is interesting to note the intimate relations between a double summable series and the corresponding double integral. Let us take the series $\overset{\infty}{S}_0 \overset{\infty}{S}_0 u_{mn}$. This is summable by rows and the results obtained, thus are also summable. Considering the corresponding double integral $\int_0^\infty \int_0^\infty e^{-x-y}\, u\,(x, y)\, dxdy$, de la Vallee Poussin's condition that $\int_0^\infty e^{-x-y}\, u\,(x, y)\, dy$ should converge, implies the property of summability of the sums of the rows. The convergence of $\int_0^\infty e^{-x-y}\, u\,(x, y)\, dx$ implies similarly the property of summability of the sums of the columns. Thus we see that, if a *summable double series were to have a unique sum, when taken by rows or columns, or as a double series, it is necessary that the corresponding double integral should satisfy the conditions of De La Vallee Poussin in virtue of which the value of the double integral is the same, even though there is a change of order of integration.*

(*To be continued.*)

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* 'Summable' is understood in the restricted sense of § 2 of this paper. Such series as $1 + 2 + 3 + \ldots\ldots\ldots\ldots$ are not contemplated in connection with this paper.

On Some Infinite Products and Series: Part II.

By M. Bhimaseña Rao and M. Venkatarama Ayyar.

[In Part I* of this paper several general results were obtained wherein a was put equal to $\pi r(\cos\theta + i\sin\theta)$ and real and imaginary parts were separated. The equated *real* parts led to the evaluation of several *theta*-products and a few series. In this part, we propose to deal with the results obtained on equating the *imaginary* parts. We add also a *direct* method of establishing a general type of relation between several of the series got. In the course of that work, a number of other results are found.]

§ 1. In result I of Part I, put

$$a = \pi r(\cos\theta + i\sin\theta),$$

and equate the imaginary parts. We get

$$-\left[\frac{e^{2\pi r\cos\theta}\sin(2\pi r\sin\theta)}{e^{4\pi r\cos\theta} - 2e^{2\pi r\cos\theta}\cos(2\pi r\sin\theta) + 1}\right.$$

$$\left. + 2^{2n+1}\frac{e^{4\pi r\cos\theta}\sin(4\pi r\sin\theta)}{e^{8\pi r\cos\theta} - 2e^{4\pi r\cos\theta}\cos(4\pi r\sin\theta) + 1} + \ldots\ldots\right]$$

$$+ (-1)^n\left[\frac{\cos(2n+2)\theta}{r^{2n+2}}\right]\times$$

$$\left\{\frac{e^{\frac{2\pi}{r}\cos\theta}\sin\left(\frac{2\pi}{r}\sin\theta\right)}{e^{\frac{4\pi}{r}\cos\theta} - 2e^{\frac{2\pi}{r}\cos\theta}\cos\left(\frac{2\pi}{r}\sin\theta\right) + 1}\right.$$

$$\left. + 2^{2n+1}\frac{e^{\frac{4\pi}{r}\cos\theta}\sin\left(\frac{4\pi}{r}\sin\theta\right)}{e^{\frac{8\pi}{r}\cos\theta} - 2e^{\frac{4\pi}{r}\cos\theta}\cos\left(\frac{4\pi}{r}\sin\theta\right) + 1} + \ldots\ldots\right\}$$

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* *Vide*: *J. I. M. S.*, Vol. XV., No. 7, pp. 150—162.

$$+ (-1)^{n+1}\left[\frac{\sin(2n+2)\theta}{r^{2n+2}}\right]\times$$

$$\left\{\frac{e^{\frac{2\pi}{r}\cos\theta}\cos\left(\frac{2\pi}{r}\sin\theta\right)-1}{e^{\frac{4\pi}{r}\cos\theta}-2e^{\frac{2\pi}{r}\cos\theta}\cos\left(\frac{2\pi}{r}\sin\theta\right)+1}+2^{2n+1}\frac{e^{\frac{4\pi}{r}\cos\theta}\cos\left(\frac{4\pi}{r}\sin\theta\right)-1}{e^{\frac{8\pi}{r}\cos\theta}-2e^{\frac{4\pi}{r}\cos\theta}\cos\left(\frac{4\pi}{r}\sin\theta\right)+1}+\ldots\ldots\right\}$$

$$=\frac{B_{n+1}}{4(n+1)}\cdot\frac{-\sin(2n+2)\theta}{r^{2n+2}} \quad \ldots \quad \ldots \text{ (A)}$$

In (A), put $r=1$ and $\sin\theta=\frac{1}{2}$, $\cos\theta=\frac{\sqrt{3}}{2}$. Then,

$$(-1)^{n+1}\sin(n+1)\frac{\pi}{3}\cdot\left[\frac{-e^{\pi\sqrt{3}}-1}{(e^{\pi\sqrt{3}}+1)^2}+2^{2n+1}\cdot\frac{e^{2\pi\sqrt{3}}-1}{(e^{2\pi\sqrt{3}}-1)^2}+\ldots\ldots\right]$$

$$=\frac{B_{n+1}}{4(n+1)}\cdot\left(-\sin(n+1)\frac{\pi}{3}\right).$$

So, when $\sin(n+1)\frac{\pi}{3}$ is not zero, *i.e.*, *when* $n \not\equiv 2 \pmod 3$,

we get

$$\frac{1^{2n+1}}{1+e^{\pi\sqrt{3}}}+\frac{2^{2n+1}}{1-e^{2\pi\sqrt{3}}}+\frac{3^{2n+1}}{1+e^{3\pi\sqrt{3}}}+\frac{4^{2n+1}}{1-e^{4\pi\sqrt{3}}}+\ldots\ldots$$

$$=(-1)^{n+1}\cdot\frac{B_{n+1}}{4(n+1)}^{*} \quad \ldots \quad \ldots \text{ (I)}$$

Again, in (1), put $r=\frac{1}{\sqrt{2}}$ and $\sin\theta=\frac{1}{\sqrt{2}}$, $\cos\theta=\frac{1}{\sqrt{2}}$. Then,

$$(-1)^{n+1}2^{n+1}\sin(n+1)\frac{\pi}{2}\left[\frac{e^{2\pi}-1}{(e^{2\pi}-1)^2}+2^{2n+1}\cdot\frac{e^{4\pi}-1}{(e^{4\pi}-1)^2}+\ldots\ldots\right]$$

$$=2^{n+1}\frac{B_{n+1}}{4(n+1)}\cdot\left(-\sin(n+1)\frac{\pi}{2}\right)$$

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* Compare with result XVIII of Part I.

When $\sin(n+1)\frac{\pi}{2}$ is not zero, *i.e.*, when *n is even*,

$$\frac{1}{e^{2\pi}-1}+\frac{2^{2n+1}}{e^{4\pi}-1}+\frac{3^{2n+1}}{e^{6\pi}-1}+\ldots\ldots\ldots$$

$$=\frac{B_{n+1}}{4(n+1)}.\dagger \quad \ldots \qquad \ldots \text{ (II-}a\text{)}$$

Once again, in (A), put $r=\sqrt{2}$, and $\sin\theta=\frac{1}{\sqrt{2}}$. Then,

$$(-1)^{n+1}.\frac{\sin(n+1)\frac{\pi}{2}}{2^{n+1}}.\left[\frac{-e^{\pi}-1}{(e^{\pi}+1)^2}+2^{3n+1}.\frac{e^{2\pi}-1}{(e^{2\pi}-1)^2}+\ldots\ldots\right]$$

$$=\frac{B_{n+1}}{4(n+1)}.\left\{-\frac{\sin(n+1)\frac{\pi}{2}}{2^{n+1}}\right\}.$$

Here also, when *n is even*,

$$\frac{1}{1+e^{\pi}}+\frac{2^{2n+1}}{1-e^{2\pi}}+\frac{3^{2n+1}}{1+e^{3\pi}}+\frac{4^{2n+1}}{1-e^{4\pi}}+\ldots\ldots$$

$$=-\frac{B_{n+1}}{4(n+1)}. \qquad \ldots \qquad \ldots \text{ (II-}b\text{)}$$

As in both cases, *n is even*, adding (II-*a*) and (II-*b*), we have

$$\operatorname{cosech}(\pi)-2^{2n+1}\operatorname{cosech}(2\pi)+3^{2n+1}\operatorname{cosech}(3\pi)$$
$$-4^{2n+1}\operatorname{cosech}(4\pi)+\ldots\ldots\ldots=0. \qquad \ldots \quad \text{(II)}$$

Again, in (A), put $r=\frac{1}{2}$ and $\sin\theta=\frac{1}{2}$, $\cos\theta=\frac{\sqrt{3}}{2}$. Then,

$$-\left\{\frac{e^{\frac{\pi\sqrt{3}}{2}}}{e^{\pi\sqrt{3}}+1}+3^{2n+1}.\frac{-e^{\frac{3\pi\sqrt{3}}{2}}}{e^{3\pi\sqrt{3}}+1}+5^{2n+1}.\frac{e^{\frac{5\pi\sqrt{3}}{2}}}{e^{5\pi\sqrt{3}}+1}+\ldots\ldots\right\}$$

$$+(-1)^{n+1}\sin(n+1)\frac{\pi}{3}\;2^{2n+2}\times$$

$$\left[\frac{e^{2\pi\sqrt{3}}-1}{(e^{2\pi\sqrt{3}}-1)^2}+2^{2n+1}.\frac{e^{4\pi\sqrt{3}}-1}{(e^{4\pi\sqrt{3}}-1)^2}+\ldots\ldots\right]$$

$$=\frac{-B_{n+1}}{4(n+1)}.\sin(n+1)\frac{\pi}{3}.2^{2n+2}.$$

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† Compare with result II of Part I.

Here, let $n = 2 \pmod 3$, making the 2nd part of the L. H. S. and the R. H. S. zero. Then, we get

$$\operatorname{sech}\left(\frac{\pi\sqrt{3}}{2}\right) - 3^{2n+1}\operatorname{sech}\left(\frac{3\pi\sqrt{3}}{2}\right) + 5^{2n+1}\operatorname{sech}\left(\frac{5\pi\sqrt{3}}{2}\right) - \ldots\ldots = 0, \qquad \ldots \text{ (III)}$$

a result parallel to a problem of the late S. Ramanujan, F. R. S., which occurs as result IV below.

§ 2. Take result IV of Part I and therein put

$$a = \pi r(\cos\theta + i\sin\theta),$$

and equate the imaginary parts. Then,

$$-\left[\frac{e^{\pi r\cos\theta}\sin(\pi r\sin\theta)}{1+2e^{\pi r\cos\theta}\cos(\pi r\sin\theta)+e^{2\pi r\cos\theta}} + 3^{2n+1}\cdot\frac{e^{3\pi r\cos\theta}\sin(3\pi r\sin\theta)}{1+2e^{3\pi r\cos\theta}\cos(3\pi r\sin\theta)+e^{6\pi r\sin\theta}} + \ldots\ldots\right]$$

$$+(-1)^n\frac{\cos(2n+2)\theta}{r^{2n+2}}\cdot\left\{\frac{e^{\frac{\pi}{r}\cos\theta}\sin\left(\frac{\pi}{r}\sin\theta\right)}{1+2e^{\frac{\pi}{r}\cos\theta}\cos\left(\frac{\pi}{r}\sin\theta\right)+e^{\frac{2\pi}{r}\cos\theta}} + 3^{2n+1}\cdot\frac{e^{\frac{3\pi}{r}\cos\theta}\sin\left(\frac{3\pi}{r}\sin\theta\right)}{1+2e^{\frac{3\pi}{r}\cos\theta}\cos\left(\frac{3\pi}{r}\sin\theta\right)+e^{\frac{6\pi}{r}\cos\theta}} + \ldots\ldots\right\}$$

$$+(-1)^{n+1}\cdot\frac{\sin(2n+2)\theta}{r^{2n+2}}\cdot\left\{\frac{1+e^{\frac{\pi}{r}\cos\theta}\cos\left(\frac{\pi}{r}\sin\theta\right)}{1+2e^{\frac{\pi}{r}\cos\theta}\cos\left(\frac{\pi}{r}\sin\theta\right)+e^{\frac{2\pi}{r}\cos\theta}} + 3^{2n+1}\frac{1+e^{\frac{3\pi}{r}\cos\theta}\cos\left(\frac{3\pi}{r}\sin\theta\right)}{1+2e^{\frac{3\pi}{r}\cos\theta}\cos\left(\frac{3\pi}{r}\sin\theta\right)+e^{\frac{6\pi}{r}\cos\theta}} + \ldots\ldots\right\}$$

$$= \frac{B_{n+1}}{4(n+1)}\cdot(2^{2n+1}-1)\cdot\left(\frac{-\sin(2n+2)\theta}{r^{2n+2}}\right). \qquad \ldots \text{ (B)}$$

In (B), put $r = \frac{1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}} = \cos\theta$. Then,

$$-\left[\frac{e^{\frac{\pi}{2}}}{1+e^{\pi}} - 3^{2n+1}\frac{e^{\frac{3\pi}{2}}}{1+e^{3\pi}} + 5^{2n+1}.\frac{e^{\frac{5\pi}{2}}}{1+e^{5\pi}} - \ldots\ldots\right]$$

$$+ 2^{n+1}.(-1)^{n+1}.\sin(n+1)\frac{\pi}{2}\left[\frac{1}{1-e^{\pi}} + \frac{3^{2n+1}}{1+e^{3\pi}} + \frac{5^{2n+1}}{1-e^{5\pi}} + \ldots\ldots\right.$$

$$= \frac{B_{n+1}}{4(n+1)}.(2^{2n+1}-1)\left\{-\sin(n+1)\frac{\pi}{2}\right\}.2^{n+1}.$$

Here, take n to be *odd*. Then, the 2nd part of the L. H. S. and the R. H. S. are equal to zero, and we get

$$\operatorname{sech}\left(\frac{\pi}{2}\right) - 3^{2n+1}\operatorname{sech}\left(\frac{3\pi}{2}\right)$$

$$+ 5^{2n+1}\operatorname{sech}\left(\frac{5\pi}{2}\right) - \ldots\ldots = 0^{*}, \qquad \ldots \quad \text{(IV)}$$

the problem of Ramanujan referred to above.

§ 3. In result IX of Part I, put

$$a = r(\cos\theta + i\sin\theta),$$

take the logarithms of both sides, separate into real and imaginary parts and equate the imaginary parts; we have

$$\left[\tan^{-1}\frac{e^{-2\pi r\cos\theta}.\sin(2\pi r\sin\theta)}{1-e^{-2\pi r\cos\theta}.\cos(2\pi r\sin\theta)}\right.$$

$$+\tan^{-1}\frac{e^{-4\pi r\cos\theta}.\sin(4\pi r\sin\theta)}{1-e^{-4\pi r\cos\theta}.\cos(4\pi r\sin\theta)} + \ldots\ldots\Big]$$

$$+\left\{\tan^{-1}\frac{e^{-\frac{2\pi}{r}\cos\theta}\sin\left(\frac{2\pi}{r}\sin\theta\right)}{1-e^{-\frac{2\pi}{r}\cos\theta}\cos\left(\frac{2\pi}{r}\sin\theta\right)}\right.$$

$$\left.+\tan^{-1}\frac{e^{-\frac{4\pi}{r}\cos\theta}\sin\left(\frac{4\pi}{r}\sin\theta\right)}{1-e^{-\frac{4\pi}{r}\cos\theta}\cos\left(\frac{4\pi}{r}\sin\theta\right)} + \ldots\ldots\right\}$$

$$= \frac{\pi.\sin\theta}{12}\left(r+\frac{1}{r}\right) - \frac{\theta}{2}. \qquad \ldots \qquad \ldots \quad \text{(C)}$$

* *Vide J. I. M. S.*, Vol. IV., No. 2, Question 358,

It can easily be seen that, in general, such values of r and θ as satisfy the equations

$$\left.\begin{aligned} 2r \sin\theta &= \frac{2n+1}{2} \\ \frac{2}{r} \sin\theta &= \frac{2m+1}{2} \end{aligned}\right\},$$

where m and n are zero or positive integers, will lead to neat results. However, we should have

$$\sin^2\theta = \frac{(2n+1)(2m+1)}{16}.$$

This condition clearly limits the number of cases. The various particular values of n and m, therefore, which would give us distinct results are

$n = 0$; $m = 0, 1, 2, 3, 4, 5, 6$ or 7;
$n = 1$; $m = 0, 1$, or 2;
$n = 2$; $m = 0$, or 1.

Even among these, three cases do not give distinct values for r and θ. Hence, the values of r that may be used are

$$r = 1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{7}}, \frac{1}{3}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{13}}, \frac{1}{\sqrt{15}}, 1 \text{ and } \frac{3}{\sqrt{15}}$$

the corresponding values for $\sin\theta$, in order, being

$$\sin\theta = \frac{1}{4}, \frac{\sqrt{3}}{4}, \frac{\sqrt{5}}{4}, \frac{\sqrt{7}}{4}, \frac{3}{4}, \frac{\sqrt{11}}{4}, \frac{\sqrt{13}}{4}, \frac{\sqrt{15}}{4}, \frac{3}{4}, \text{ and } \frac{\sqrt{15}}{4}.$$

Among the above sets of values, two, *viz.*,

$$r = 1, \sin\theta = \tfrac{1}{4}$$
$$r = 1, \sin\theta = \tfrac{3}{4}$$

lead immediately to two single series and the remaining 8 sets to sum or difference of two series.

If, in (C), we put $r = 1$, $\sin\theta = \frac{1}{4}$, and $\cos\theta = \frac{\sqrt{15}}{4}$, then

$$2\{\tan^{-1} e^{-\frac{\pi}{2}\sqrt{15}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{15}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{15}} - \ldots\}$$
$$= \frac{\pi}{48} \cdot 2 \cdot -\frac{1}{2}\sin^{-1}\frac{1}{4},$$

or $$\tan^{-1} e^{-\frac{\pi}{2}\sqrt{15}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{15}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{15}} - \ldots\ldots$$
$$= \frac{\pi}{48} - \frac{1}{4}\tan^{-1}\frac{1}{\sqrt{15}}. \qquad \ldots \text{ (V)}$$

Similarly, if we put $r = 1$, $\sin\theta = \frac{3}{4}$, $\cos\theta = \frac{\sqrt{7}}{4}$, we have

$$\tan^{-1} e^{-\frac{\pi}{2}\sqrt{7}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{7}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{7}} - \ldots\ldots$$

$$= \frac{1}{4}\tan^{-1}\frac{3}{\sqrt{7}} - \frac{\pi}{16}. \quad \ldots \quad \text{(VI)}$$

Proceeding in an exactly similar way with the remaining sets of values, we obtain the following :—

$$\left[\tan^{-1} e^{-\frac{\pi}{2}\sqrt{\frac{13}{3}}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{\frac{13}{3}}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{\frac{13}{3}}} - \ldots.\right]$$

$$- \left[\tan^{-1} e^{-\frac{\pi}{2}\sqrt{13.3}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{13.3}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{13.3}} - \ldots.\right]$$

$$= \frac{\pi}{12} - \frac{1}{2}\tan^{-1}\sqrt{\frac{3}{13}}. \quad \ldots \quad \text{(VII)}$$

$$\left[\tan^{-1} e^{-\frac{\pi}{2}\sqrt{\frac{11}{5}}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{\frac{11}{5}}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{\frac{11}{5}}} - \ldots.\right]$$

$$+ \left[\tan^{-1} e^{-\frac{\pi}{2}\sqrt{11.5}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{11.5}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{11.5}} - \ldots.\right]$$

$$= \frac{\pi}{8} - \frac{1}{2}\tan^{-1}\sqrt{\frac{5}{11}}. \quad \ldots \quad \ldots \quad \text{(VIII)}$$

$$\left[\tan^{-1} e^{-\frac{\pi}{2}\cdot\frac{3}{\sqrt{7}}} - \tan^{-1} e^{-\frac{3\pi}{2}\cdot\frac{3}{\sqrt{7}}} + \tan^{-1} e^{-\frac{5\pi}{2}\cdot\frac{3}{\sqrt{7}}} - \ldots.\right]$$

$$- \left[\tan^{-1} e^{-\frac{\pi}{2}.3\sqrt{7}} - \tan^{-1} e^{-\frac{3\pi}{2}.3\sqrt{7}} + \tan^{-1} e^{-\frac{5\pi}{2}3\sqrt{7}} - \ldots.\right]$$

$$= \frac{\pi}{6} - \tfrac{1}{2}\tan^{-1}\left(\frac{\sqrt{7}}{3}\right). \quad \ldots \quad \ldots \quad \text{(IX)}$$

$$\left[\tan^{-1} e^{-\frac{\pi}{2}\cdot\frac{\sqrt{7}}{3}} - \tan^{-1} e^{-\frac{3\pi}{2}\cdot\frac{\sqrt{7}}{3}} + \tan^{-1} e^{-\frac{5\pi}{2}\frac{\sqrt{7}}{3}} - \ldots.\right]$$

$$+ \left[\tan^{-1} e^{-\frac{\pi}{2}.3\sqrt{7}} - \tan^{-1} e^{-\frac{3\pi}{2}.3\sqrt{7}} + \tan^{-1} e^{-\frac{5\pi}{2}.3\sqrt{7}} - \ldots.\right]$$

$$= \frac{5\pi}{24} - \frac{1}{2}\tan^{-1}\left(\frac{3}{\sqrt{7}}\right) \quad \ldots \quad \ldots \quad \text{(X)}$$

$$\left[\tan^{-1} e^{-\frac{\pi}{2}\sqrt{\frac{5}{11}}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{\frac{5}{11}}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{\frac{5}{11}}} - \ldots\right]$$

$$-\left[\tan^{-1} e^{-\frac{\pi}{2}\sqrt{5.11}} - \tan^{-1} e^{-\frac{3\pi}{2}\cdot\sqrt{5.11}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{5.11}} - \ldots\right]$$

$$= \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{\sqrt{11}}{5}\right). \quad \ldots \quad \ldots \quad \text{(XI)}$$

$$\left[\tan^{-1} e^{-\frac{\pi}{2}\sqrt{\frac{3}{13}}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{\frac{3}{13}}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{\frac{3}{13}}} - \ldots\right]$$

$$-\left[\tan^{-1} e^{-\frac{\pi}{2}\sqrt{13.3}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{13.3}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{13.3}} - \ldots\right]$$

$$= \frac{7\pi}{24} - \frac{1}{2}\tan^{-1}\left(\frac{\sqrt{13}}{3}\right). \ldots \quad \ldots \quad \text{(XII)}$$

$$\left[\tan^{-1} e^{-\frac{\pi}{2}\cdot\frac{1}{\sqrt{15}}} - \tan^{-1} e^{-\frac{3\pi}{2}\cdot\frac{1}{\sqrt{15}}} + \tan^{-1} e^{-\frac{5\pi}{2}\cdot\frac{1}{\sqrt{15}}} - \ldots\right]$$

$$-\left[\tan^{-1} e^{-\frac{\pi}{2}\sqrt{15}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{15}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{15}} - \ldots\right]$$

$$= \frac{\pi}{3} - \frac{1}{2}\tan^{-1}(\sqrt{15}). \quad \ldots \quad \ldots \quad \text{(XIII)}$$

$$\left[\tan^{-1} e^{-\frac{\pi}{2}\cdot\frac{3}{\sqrt{15}}} - \tan^{-1} e^{-\frac{3\pi}{2}\cdot\frac{3}{\sqrt{15}}} + \tan^{-1} e^{-\frac{5\pi}{2}\cdot\frac{3}{\sqrt{15}}} - \ldots\right]$$

$$-\left[\tan^{-1} e^{-\frac{\pi}{2}\cdot\frac{\sqrt{15}}{3}} - \tan^{-1} e^{-\frac{3\pi}{2}\cdot\frac{\sqrt{15}}{3}} + \tan^{-1} e^{-\frac{5\pi}{2}\cdot\frac{\sqrt{15}}{3}} - \ldots\right]$$

$$= \frac{1}{2}\tan^{-1}(\sqrt{15}) - \frac{\pi}{6}. \quad \ldots \quad \ldots \quad \text{(XIV)}$$

We could also try a different set of values of r and θ, namely, those that satisfy

$$2r\sin\theta = \frac{2n+1}{2} \text{ and } \frac{2}{r}\sin\theta = m.$$

where m and n are zero or positive integers. Of the various values of r and θ to which these lead, some are useful. When

$$r = \frac{1}{2}, \ \frac{1}{\sqrt{6}}, \text{ or } \frac{1}{\sqrt{8}},$$

and

$$\sin\theta = \frac{1}{2}, \ \sqrt{\frac{3}{8}}, \text{ or } \frac{1}{\sqrt{2}},$$

we obtain

$$\tan^{-1} e^{-\frac{\pi}{2}\sqrt{3}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{3}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{3}} - \ldots\ldots$$

$$= \frac{\pi}{48}. \quad \ldots \quad \ldots \quad \text{(XV)}$$

$$\tan^{-1} e^{-\frac{\pi}{2}\cdot\frac{\sqrt{15}}{3}} - \tan^{-1} e^{-\frac{3\pi}{2}\cdot\frac{\sqrt{15}}{3}} + \tan^{-1} e^{-\frac{5\pi}{2}\cdot\frac{\sqrt{15}}{3}} - \ldots\ldots$$

$$= \frac{\pi}{48} - \frac{1}{2}\tan^{-1}\sqrt{\frac{3}{5}}. \quad \ldots \quad \text{(XVI)}$$

$$\tan^{-1} e^{-\frac{\pi}{2}} - \tan^{-1} e^{-\frac{3\pi}{2}} + \tan^{-1} e^{-\frac{5\pi}{2}} - \ldots\ldots = \frac{\pi}{16}. \quad \text{(XVII)}$$

Certain other values of r and θ, *viz.*,

$$r = \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{12}} \text{ and } \frac{1}{\sqrt{14}},$$

$$\sin\theta = \sqrt{\tfrac{5}{8}}, \sqrt{\tfrac{3}{4}}, \text{ and } \sqrt{\tfrac{7}{8}};$$

lead to the following :—

$$\tan^{-1} e^{-\frac{\pi}{2}\cdot\frac{3}{\sqrt{15}}} - \tan^{-1} e^{-\frac{3\pi}{2}\cdot\frac{3}{\sqrt{15}}} + \tan^{-1} e^{-\frac{5\pi}{2}\cdot\frac{3}{\sqrt{15}}} - \ldots\ldots$$

$$= \frac{11\pi}{48} - \frac{1}{2}\tan^{-1}\sqrt{\frac{5}{3}}. \quad \ldots \quad \text{(XVIII)}$$

$$\tan^{-1} e^{-\frac{\pi}{2}\cdot\frac{1}{\sqrt{3}}} - \tan^{-1} e^{-\frac{3\pi}{2}\cdot\frac{1}{\sqrt{3}}} + \tan^{-1} e^{-\frac{5\pi}{2}\cdot\frac{1}{\sqrt{3}}} - \ldots\ldots$$

$$= \frac{5\pi}{48} \quad \ldots \quad \ldots \quad \text{(XIX)}$$

and

$$\tan^{-1} e^{-\frac{\pi}{2}\cdot\frac{1}{\sqrt{7}}} - \tan^{-1} e^{-\frac{3\pi}{2}\cdot\frac{1}{\sqrt{7}}} + \tan^{-1} e^{-\frac{5\pi}{2}\cdot\frac{1}{\sqrt{7}}} - \ldots\ldots$$

$$= \frac{5\pi}{16} - \frac{1}{2}\tan^{-1}(\sqrt{7}). \quad \ldots \quad \text{(XX)}$$

(It may be noticed in passing that results XVI and XVIII give separately the values of the two parts in result XIV.)

§ 4. Now, let us take the result 8 of Part I ; therein put

$$a = r(\cos\theta + i\sin\theta),$$

take the logarithms of both sides, separate into real and imaginary parts and then equate the imaginary parts. We obtain

$$\left[\tan^{-1}\frac{e^{-\pi r\cos\theta}}{1+e^{-\pi r\cos\theta}}\frac{\sin(\pi r\sin\theta)}{\cos(\pi r\sin\theta)}\right.$$

$$\left.+\tan^{-1}\frac{e^{-3\pi r\cos\theta}.\sin(3\pi r\sin\theta)}{1+e^{-3\pi r\cos\theta}\cos(3\pi r\sin\theta)}+\ldots\ldots\right]$$

$$+\left\{\tan^{-1}\frac{e^{-\frac{\pi}{r}\cos\theta}.\sin\left(\frac{\pi}{r}\sin\theta\right)}{1+e^{-\frac{\pi}{r}\cos\theta}\cos\left(\frac{\pi}{r}\sin\theta\right)}\right.$$

$$\left.+\tan^{-1}\frac{e^{-\frac{3\pi}{r}\cos\theta}\sin\left(\frac{3\pi}{r}\sin\theta\right)}{1+e^{-\frac{3\pi}{r}\cos\theta}\left(\cos\frac{3\pi}{r}\sin\theta\right)}+\ldots\ldots'\right\}$$

$$=\frac{\pi\sin\theta}{24}.\left(r+\frac{1}{r}\right)\qquad\ldots\quad(D)$$

Here, values of r and θ which make

$$2r\sin\theta=2n+1,$$

and

$$\frac{2}{r}\sin\theta=2m+1$$

lead to neat results. Among them are

$$r=1,\ \frac{1}{\sqrt{3}}\text{ and }\sqrt{3},$$

$$\sin\theta=\tfrac{1}{2},\ \frac{\sqrt{3}}{2}\text{ and }\frac{\sqrt{3}}{2}.$$

Of these, $r=1,\ \sin\theta=\frac{1}{2}$ give

$$\tan^{-1}e^{-\frac{\pi}{2}\sqrt{3}}-\tan^{-1}e^{-\frac{3\pi}{2}\sqrt{3}}+\tan^{-1}e^{-\frac{5\pi}{2}\sqrt{3}}-\ldots\ldots=\frac{\pi}{48},$$

which is the result XV already got.

Both $r = \frac{1}{\sqrt{3}}$ and $r = \sqrt{3}$ with $\sin\theta = \frac{\sqrt{3}}{2}$ lead to the same series, *viz* :—

$$\left[\tan^{-1} e^{-\frac{\pi}{2\sqrt{3}}} - \tan^{-4} e^{-\frac{3\pi}{2\sqrt{3}}} + \tan^{-1} e^{-\frac{5\pi}{2\sqrt{3}}} - \ldots\ldots\right]$$

$$-\left[\tan^{-1} e^{-\frac{\pi}{2}\sqrt{3}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{3}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{3}} - \ldots\ldots\right]$$

$$= \frac{\pi}{12},$$

not, however, a new result as can be seen on taking the difference between results XIX and XV above.

§ 5. We shall now derive in a direct manner a very important result connected with the above type of series. The result is occasionally useful in evaluating some series among the above.

It is well-known that

$$\int_0^\infty \frac{2\cos 2xt}{\cosh \pi t}\, dt = \operatorname{sech} x.*$$

$$\therefore \quad \int_0^\infty \frac{2\sin xt \sin at}{\cosh \pi t}\, dt = \tfrac{1}{2}\left[\operatorname{sech}\frac{x-a}{2} - \operatorname{sech}\frac{x+a}{2}\right]$$

Now,
$$\int_0^\infty \frac{2\sin at}{\cosh \pi t}\, \{2\sin xt - 2\sin 3xt + \ldots\ldots + 2\sin(4n+1)xt\}\, dt$$

$$= \int_0^\infty \frac{2\sin at}{\cosh \pi t} \cdot \frac{\sin(2n+1)2xt}{\cos xt}\, dt. \quad \ldots \quad \text{(E)}$$

On the L. H. S. of (E), integrate term by term and let n tend to infinity. Then we have

$$\left(\operatorname{sech}\frac{x-a}{2} - \operatorname{sech}\frac{3x-a}{2} + \operatorname{sech}\frac{5x-a}{2} - \ldots\ldots\right)$$

$$-\left(\operatorname{sech}\frac{x+a}{2} - \operatorname{sech}\frac{3x+a}{2} + \operatorname{sech}\frac{5x+a}{2} - \ldots\ldots\right). \quad \text{(F)}$$

---

* *Vide* Bromwich: *Introduction to Infinite Series*, p. 472, Ex. 26.

Again, taking the R.H.S.,

$$\lim_{n\to\infty} \int \frac{\sin(2n+1)\,2xt}{\sin 2xt} \cdot \frac{2\sin xt\; 2\sin at}{\cosh \pi t}\, dt$$

becomes, on changing t into $\frac{t}{2x}$,

$$\lim_{n\to\infty} \int_0^\infty \frac{\sin(2n+1)\,t}{\sin t} \cdot \left\{ \frac{\sin\frac{t}{2} \cdot 2 \cdot \sin\left(\frac{at}{2x}\right)}{x\cosh\left(\frac{\pi t}{2x}\right)} \right\} dt$$

which on using the result of Ex. 4 on p. 447 of Bromwich's *Introduction to Infinite Series*, gives

$$\frac{\pi}{2} \cdot \left\{ \frac{4}{x} \frac{\sin\left(\frac{a\pi}{2x}\right)}{\cosh\left(\frac{\pi^2}{2x}\right)} - \frac{4}{x} \cdot \frac{\sin\left(\frac{3a\pi}{2x}\right)}{\cosh\left(\frac{3\pi^2}{2x}\right)} + \ldots \right\} \qquad \text{(G)}$$

Equating (F) and (G),

$$\left[\operatorname{sech}\frac{x-a}{2} - \operatorname{sech}\frac{3x-a}{2} + \operatorname{sech}\frac{5x-a}{2} - \ldots\ldots\right]$$

$$-\left[\operatorname{sech}\frac{x+a}{2} - \operatorname{sech}\frac{3x+a}{2} + \operatorname{sech}\frac{5x+a}{2} - \ldots\ldots\right]$$

$$= \frac{2\pi}{x} \cdot \left\{ \frac{\sin\left(\frac{a\pi}{2x}\right)}{\cosh\left(\frac{\pi^2}{2x}\right)} - \frac{\sin\left(\frac{3a\pi}{2x}\right)}{\cosh\left(\frac{3\pi^2}{2x}\right)} + \ldots\ldots \right\} \qquad \text{(XXI)}$$

Put $a = x$ in the above and transpose. We get

$$(\operatorname{sech} x - \operatorname{sech} 2x + \operatorname{sech} 3x - \ldots\ldots)$$

$$+ \frac{\pi}{x}\left(\operatorname{sech}\frac{\pi^2}{2x} - \operatorname{sech}\frac{3\pi^2}{2x} + \operatorname{sech}\frac{5\pi^2}{2x} - \ldots\ldots\right) = \frac{1}{2}. \qquad \text{(XXII)}$$

§ 6. Now, expand each term on the L.H.S. of XXI. Then

$$2\left[e^{-\frac{x-a}{2}} - e^{-3.\frac{x-a}{2}} + e^{-5.\frac{x-a}{2}} - \ldots\ldots\right]$$

$$-2\left[e^{-\frac{3x-a}{2}} - e^{-3\frac{3x-a}{2}} + e^{-5.\frac{3x-a}{2}} - \ldots\ldots\right]$$

$$+2\left[e^{-\frac{5x-a}{2}} - e^{-3.\frac{5x-a}{2}} + e^{-5.\frac{5x-a}{2}} - \ldots\ldots\right]$$

$$- \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-2\left[e^{-\frac{x+a}{2}} - e^{-3.\frac{x+a}{2}} + e^{-5.\frac{x+a}{2}} - \ldots\ldots\right]$$

$$+2\left[e^{-\frac{3x+a}{2}} - e^{-3.\frac{3x+a}{2}} + e^{-5.\frac{5x+a}{2}} - \ldots\ldots\right]$$

$$-2\left[e^{-\frac{5x+a}{2}} - e^{-3.\frac{5x+a}{2}} + e^{-5.\frac{5x+3}{2}} - \ldots\ldots\right]$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \frac{2\pi}{x}.\left[\sin\left(\frac{a\pi}{2x}\right)\operatorname{sech}\left(\frac{\pi^2}{2x}\right) - \sin\left(\frac{3a\pi}{2x}\right)\operatorname{sech}\left(\frac{3\pi^2}{2x}\right)\right.$$

$$\left. + \sin\left(\frac{5a\pi}{2x}\right)\operatorname{sech}\left(\frac{5\pi^2}{2x}\right) - \ldots\ldots\right].$$

Adding each set on the L. H. S. column by column,

$$\left[e^{\frac{a}{2}}\operatorname{sech}\left(\frac{x}{2}\right) - e^{\frac{3a}{2}}\operatorname{sech}\left(\frac{3x}{2}\right) - e^{\frac{5a}{2}}.\operatorname{sech}\left(\frac{5x}{2}\right) - \ldots\ldots\right]$$

$$-\left[e^{-\frac{a}{2}}\operatorname{sech}\left(\frac{x}{2}\right) - e^{-\frac{3a}{2}}\operatorname{sech}\left(\frac{3x}{2}\right) + e^{-\frac{5a}{2}}\operatorname{sech}\left(\frac{5x}{2}\right) - \ldots\ldots\right]$$

$$= \frac{2\pi}{x}\left[\sin\left(\frac{a\pi}{2x}\right)\operatorname{sech}\left(\frac{\pi^2}{2x}\right) - \sin\left(\frac{3a\pi}{x}\right)\operatorname{sech}\left(\frac{3\pi^2}{2x}\right)\right.$$

$$\left. + \sin\left(\frac{5a\pi}{2x}\right)\operatorname{sech}\left(\frac{5\pi^2}{2x}\right) - \ldots\ldots\right].$$

Or,

$$\sinh\left(\frac{a}{2}\right)\operatorname{sech}\left(\frac{x}{2}\right) - \sinh\left(\frac{3a}{2}\right)\operatorname{sech}\left(\frac{3x}{2}\right)$$

$$+ \sinh\left(\frac{5a}{2}\right)\operatorname{sech}\left(\frac{5x}{2}\right) - \ldots\ldots$$

$$= \frac{\pi}{x}.\left[\sin\left(\frac{a\pi}{2x}\right)\operatorname{sech}\left(\frac{\pi^2}{2x}\right) - \sin\left(\frac{3a\pi}{2x}\right)\operatorname{sech}\left(\frac{3\pi^2}{2x}\right)\right.$$

$$\left. + \sin\left(\frac{5a\pi}{2x}\right)\operatorname{sech}\left(\frac{5\pi^2}{2x}\right) - \ldots\ldots\right]. \quad \ldots \quad \text{(XXIII)}$$

§ 7. Here, equate the co-efficients of a^{2n+1} on both sides. We have

$$\frac{1}{2^{2n+1}.(2n+1)!}\cdot\left\{\operatorname{sech}\left[\frac{x}{2}\right]-3^{2n+1}.\operatorname{sech}\left[\frac{3x}{2}\right]+\right.$$

$$\left.5^{2n+1}.\operatorname{sech}\left[\frac{5x}{2}\right]-.....\right\}$$

$$=\frac{\pi}{x}\cdot\frac{(-1)^n}{(2n+1)!}\cdot\frac{\pi^{2n+1}}{x^{2n+1}.2^{2n+1}}\times$$

$$.\left\{\operatorname{sech}\left[\frac{\pi^2}{2x}\right]-3^{2n+1}\operatorname{sech}\left[\frac{3\pi^2}{2x}\right]+.....\right\}$$

Changing x into πx,

$$\operatorname{sech}\left[\frac{\pi x}{2}\right]-3^{2n+1}.\operatorname{sech}\left[\frac{3\pi x}{2}\right]+.....$$

$$=\frac{(-1)^n}{x^{2n+1}}\cdot\left\{\operatorname{sech}\left[\frac{\pi}{2x}\right]-3^{2n+1}.\operatorname{sech}\left[\frac{3\pi}{2x}\right]+.....\right\}. \quad \text{(XXIV)}$$

If in the above, x is put equal to 1 and n is taken to be odd, we obtain

$$\operatorname{sech}\left[\frac{\pi}{2}\right]-3^{2n+1}\operatorname{sech}\left[\frac{3\pi}{2}\right]+.....=0,$$

a result already obtained as result IV though in a different manner.

§ 8. In XXIV, put $n=0$. Then

$$\operatorname{sech}\left[\frac{\pi x}{2}\right]-3\operatorname{sech}\left[\frac{3\pi x}{2}\right]+.....$$

$$=\frac{1}{x^2}\left\{\operatorname{sech}\left[\frac{\pi}{2x}\right]-3\operatorname{sech}\left[\frac{3\pi}{2x}\right]+.....\right\}.$$

Multiply both sides of the above by $-\frac{\pi}{4}$ and integrate with respect to x. Then,

$$\tan^{-1}\left[e^{-\frac{\pi x}{2}}\right]-\tan^{-1}\left[e^{-\frac{3\pi x}{2}}\right]+.....$$

$$=-\tan^{-1}\left[e^{-\frac{\pi}{2x}}\right]+\tan^{-1}\left[e^{-\frac{3\pi}{2x}}\right]-.....+\text{a constant.}$$

Putting $x = 0$, the constant is found to be equal to $\frac{\pi}{8}$.

Therefore, we get

$$\left\{\tan^{-1}\left[e^{-\frac{\pi x}{2}}\right] - \tan^{-1}\left[e^{-\frac{3\pi x}{2}}\right] + \ldots\ldots\right\}$$

$$+\left\{\tan^{-1}\left[e^{-\frac{\pi}{2x}}\right] - \tan^{-1}\left[e^{-\frac{3\pi}{2x}}\right] + \ldots\ldots\right\}$$

$$= \frac{\pi}{8}.^{*} \qquad \ldots \quad \text{(XXV)}$$

With this result, we could separate into single series results like XIV and evaluate the related series if only one of the two parts above are known in value. If, in this, we put $x = 1$, we get

$$\tan^{-1}\left[e^{-\frac{\pi}{2}}\right] - \tan^{-1}\left[e^{-\frac{3\pi}{2}}\right] + \tan^{-1}\left[e^{-\frac{5\pi}{2}}\right]$$

$$- \ldots\ldots = \frac{\pi}{16},$$

a result otherwise obtained as result XVII.

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* *Cp.* Dr. Glaisher's result in Vol. 5 of the *Messenger of Mathematics.*

Summable Double Series.

By V. Thiruvenkatacharya, M.A., L.T.

[*Continued from Vol. XV, No. 10, p. 232.*]

6. Relations between

$$\sum_0^\infty \sum_0^\infty u_{mn},\ \sum_1^\infty \sum_0^\infty u_{mn},\ \sum_0^\infty \sum_1^\infty u_{mn} \text{ and } \sum_1^\infty \sum_1^\infty u_{mn}$$

Let
$$S = \sum_0^\infty \sum_0^\infty u_{mn} = \int_0^\infty \int_0^\infty e^{-x-y}\, u(x, y)\, dxdy, \quad \ldots \quad 6{\cdot}1$$

$$S_x = \sum_1^\infty \sum_0^\infty u^{mn} = \int_0^\infty \int_0^\infty e^{-x-y} \left[\frac{\partial}{\partial x} u(x, y) \right] dxdy, \quad \ldots \quad 6{\cdot}2$$

$$S_y = \sum_0^\infty \sum_1^\infty u_{mn} = \int_0^\infty \int_0^\infty e^{-x-y} \left[\frac{\partial}{\partial y} u(x, y) \right] dxdy, \quad \ldots \quad 6{\cdot}3$$

and
$$S_{xy} = \sum_1^\infty \sum_1^\infty u_{mn} = \int_0^\infty \int_0^\infty e^{-x-y} \left[\frac{\partial^2}{\partial x \partial y} u(x, y) \right] dxdy. \quad 6{\cdot}4$$

We have to justify the term by term differentiation of $u(x, y)$ with respect to x and y. It has been already supposed that the u's are such that $u(x, y)$ is convergent in Pringsheim's sense for all values of x and y; and $u(x, y)$ is a double power series in x and y, the associated radii of convergence of which are as large as possible. Therefore the double power series converges uniformly and can be differentiated any number of times with respect to either x or y or both.*

To establish the relations between S, S_x, S_y and S_{xy}, let us take S_x first and integrate it by parts.

$$S_x = \int_0^\infty \left[e^{-x-y}\, u(x, y) \right]_{x=0}^{x=\infty} dy + \int_0^\infty \int_0^\infty e^{-x-y}\, u(x, y)\, dxdy$$

$$= \int_0^\infty \left[e^{-x-y}\, u(x, y) \right]_{x=0}^{x=\infty} dy + S. \quad \ldots \quad \ldots \quad 6{\cdot}5$$

Equation 6·5 shows that if S and S_x exist, then

$$\lim_{x \to \infty} e^{-x-y}\, u(x, y)$$

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* *Vide* Goursat and Hedrick : *Mathematical Analysis*, Vol. I, p, 395.

must exist and be zero since otherwise, the integral $\int_0^\infty e^{-x-y}\, u(x, y)\, dx$ will not converge; but this is necessary for the convergence* and the uniqueness of the double integral S. It also follows from the same equation that $\int_0^\infty e^{-y}\, u(0, y)\, dy$ must converge; but $\int_0^\infty e^{-y}\, u(0, y)\, dy$ is the sum of the first row and the convergence of this integral implies that the row should be summable. Hence provided that the integrals S and S_x are convergent and the first row is summable.

$$S_x = -\int_0^\infty e^{-y}\, u\,(o, y)\, dy + S$$

i.e.,
$$S - S_x = \int_0^\infty e^{-y}\, u\,(o, y)\, dy = \text{sum of the first row.} \qquad 6{\cdot}6$$

Integrating S_y by parts, we have

$$S - S_y = \int_0^\infty e^{-x}\, u\,(x, o)\, dx = \text{sum of the first column,} \qquad 6{\cdot}7$$

provided the integrals S and S_y are convergent and the first column is summable,.

Integrating S_{xy} by parts, both with respect to x and y, we have

$$S_{xy} = u\,(o, o) - \int_0^\infty e^{-y}\, u\,(o, y)\, dy - \int_0^\infty e^{-x}\, u\,(x, o)\, dx + S. \qquad 6{\cdot}8$$

$\therefore\ S - S_{xy} =$ sum of first row + sum of first column $- u\,(o, o)$.

But $u\,(o, o) = u_{00}$, the initial term of the double series.

$\therefore\ S - S_{xy} =$ sum of first row + sum of first column $- u_{00}$. $\qquad 6{\cdot}9$

It will be seen from this that *summable double series possess properties in this respect analogous to convergent series.*

Again, the convergence of the double integral

$$\int_0^\infty \int_0^\infty e^{-x-y}\ u\,(x, y)\, dxdy$$

does not ensure the existence of a limit for $e^{-x-y}\, u\,(x, y)$ when either x or y or both tend to infinity. † *Therefore if S exists, it does not follow from this that either S_x or S_y or S_{xy} exists.*

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* Throughout this article, it is assumed that the infinite double integrals satisfy de la Valee Poussin's conditions, also, whenever they are said to be convergent.

† To see that it is possible, in general, let us use a graphical method. Consider a mountainous tract of country with a number of peaks steadily increasing in heights, the sea-level representing the xy plane. Then it is quite possible to

7. Although the convergence of $\int_0^\infty \int_0^\infty e^{-x-y}\, u\,(x, y)\, dxdy$ does not imply the existence of a limit for $e^{-x-y}\, u\,(x, y)$ as x or y or both tend to infinity, yet we can prove that $\lim e^{-x-y}\, u\,(x, y) = 0$ as x and $y \to \infty$ when $\int_0^\infty \int_0^\infty e^{-x-y} \frac{\partial}{\partial x} u\,(x, y)\, dxdy$ is convergent*, and from this we can show that

$$\int_0^\infty \int_0^\infty e^{-x-y}\, u\,(x, y)\, dxdy$$

is convergent.

For, let $$v = \int_0^y \int_0^x e^{-x-y}\, u\,(x, y)\, dxdy$$

Then $$\frac{\partial^2 v}{\partial x\, \partial y} = e^{-x-y}\, u(x, y) \qquad 7{\cdot}1$$

and $$\frac{\partial^3 v}{\partial^2 x\, \partial y} = e^{-x-y} \frac{\partial}{\partial x} u\,(x, y) - e^{-x-y}\, u\,(x, y) \qquad 7{\cdot}2$$

$$\therefore \quad e^{-x-y} \frac{\partial}{\partial x} u\,(x, y) = \frac{\partial^3 v}{\partial^2 x\, \partial y} + e^{-x-y}\, u\,(x, y)$$

and consequently integrating both the sides,

$$\int_0^y \int_0^x e^{-x-y} \frac{\partial}{\partial x}\, u\,(x, y)\, dxdy = \int_0^y \int_0^x e^{-x-y}\, u\,(x, y)\, dxdy + \frac{\partial v}{\partial x} - \int_0^y e^{-y}\, u\,(0, y)\, dy$$

---

suppose that the thickness of the peaks correspondingly decreases in such a way that the cubical contents of each of these peaks form a convergent double series satisfying Pringsheim's conditions and consequently $\int^\infty \int^\infty f\,(xy)\, dx\, dy$ may converge. For a similar illustration in the case of single infinite integrals, *vide*: Bromwich: *Infinite Series*, page 422.

* It is easy to show that $e^{-x-y}\, u\,(x, y)$ tends to zero as x tends to infinity, if it is given that

$$\int_0^\infty \int_0^\infty e^{-x-y} \frac{\partial}{\partial x}\, u\,(x, y)\, dxdy$$

is convergent. For, when this integral is assumed to be convergent, it is implied that the integral satisfies de la Valee Poussin's conditions. Hence

$$\int_0^\infty e^{-x-y} \frac{\partial}{\partial x}\, u\,(x, y)\, dx$$

must be convergent. Therefore it follows that $e^{-x-y}\, u\,(x, y)$ tends to zero as x tends to infinity (*vide* Bromwich: *Infinite Series*, Art. 101). The case when y tends to infinity can be dealt with similarly.

$$i.\,e.,\ \int_0^y \int_0^x e^{-x-y} \frac{\partial}{\partial x}\, u\,(x, y)\, dxdy = v + \frac{\partial v}{\partial x} - \int_0^y e^{-y}\, u\,(0, y)\, dy. \quad 7\cdot3$$

Then provided $\int_0^y e^{-y}\, u\,(0, y)\, dy$ is convergent as y tends to infinity, the expression on the right must tend to a definite limit when x and y tend to infinity since we have assumed that

$$\int_0^\infty \int_0^\infty e^{-x-y} \frac{\partial}{\partial x}\, u\,(x, y)\, dxdy$$

is convergent and satisfies de la Valee Poussin's conditions.

Let the limit be l and let us write

$$w = v - \int_0^\infty e^{-y}\, u\,(0, y)\, dy - l, \text{ and } \frac{\partial w}{\partial x} = \frac{\partial v}{\partial x}.$$

It follows then that $\left| w + \frac{\partial w}{\partial x} \right|$ tends to the limit zero, when x and $y \to \infty$. Thus we can chose X and Y so that

$$\left| w + \frac{\partial w}{\partial x} \right| < \varepsilon, \text{ if } x > \mathrm{X} \text{ and } y > \mathrm{Y}.$$

If w has any number of extreme values considered as a function of x, $\frac{\partial w}{\partial x}$ vanishes at each of these. Hence at any extreme value beyond X and Y we have $|\, w \,| < \varepsilon$, thus $|\, w \,| < \varepsilon$, if $x > \mathrm{X}$ and $y > \mathrm{Y}$.

$$\therefore \quad \lim_{x, y \to \infty} w = 0 \text{ and hence } \lim_{x, y \to \infty} \frac{\partial w}{\partial x} = 0.$$

But if w has not an unlimited number of extreme values as function of x, $\frac{\partial w}{\partial x}$ must be of a constant sign finally, and so w must approach a definite limit k and hence $\frac{\partial w}{\partial x}$ must approach the limit $-k$.

But if $\frac{\partial w}{\partial x}$ approaches the limit $-k$, then $\frac{w}{x}$ approaches the same limit. Thus we have $\lim w = k$; and $\lim \frac{w}{x} = -k$; but w and $\frac{w}{x}$ have the same sign and therefore k must be zero.

Hence $\lim_{x,y \to \infty} w = 0$ and $\lim_{x,y \to \infty} \frac{\partial w}{\partial x} = 0$;

or $$\int_0^\infty \int_0^\infty e^{-x-y} u(x,y)\, dx\, dy = \int_0^\infty e^{-y} u(0,y)\, dy + \int_0^\infty \int_0^\infty e^{-x-y} \left[\frac{\partial}{\partial x} u(x,y)\right] dx dy.$$

or $$\sum_0^\infty \sum_0^\infty u_{m,n} = \sum_0^\infty u_{0,n} + \sum_0^\infty \sum_0^\infty u_{m,n}. \quad \ldots \quad 7{\cdot}4$$

This shows that if $\sum_1^\infty \sum_0^\infty$ *is summable and the single series* $\sum_0^\infty u_{0,n}$ *is summable, then* $\sum_0^\infty \sum_0^\infty u_{m,n}$ *is summable; and that the converse is not true.*

Similar results can be arrived at by assuming that the integrals for S_y and S_{xy} are convergent and satisfy de la Valee Poussin's conditions.

The results obtained can be summarised as follows:—*To a summable double series any number of summable rows or columns may be added, without destroying the property of summability of the double series; but the removal of either a single row or column may destroy this property.*

8. In article 4 of this paper some examples of summable double series were given and their sums were found. Let us take some more double series the terms of which are circular functions.

(1) Let $c =$
$$\begin{aligned} &1 + \cos\theta + \cos 2\theta + \cos 3\theta + \ldots \\ &+ \cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta + \ldots \\ &+ \cos 2\theta + \cos 3\theta + \cos 4\theta + \cos 5\theta + \ldots \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

and $s =$
$$\begin{aligned} &0 + \sin\theta + \sin 2\theta + \sin 3\theta + \ldots \\ &+ \sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta + \ldots \\ &+ \sin 2\theta + \sin 3\theta + \sin 4\theta + \sin 5\theta + \ldots \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Then
$$c + i\,s = 1 + e^{i\theta} + e^{2i\theta} + \ldots\ldots\ldots\ldots$$
$$+ e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \ldots\ldots\ldots\ldots$$
$$+ e^{2i\theta} + e^{3i\theta} + e^{4i\theta} + \ldots\ldots\ldots\ldots$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The associated function $u(x, y)$ is found to be $e^{(x+y)\,e^{i\theta}}$.

$$\therefore \quad c + is = \int_0^\infty \int_0^\infty e^{-x(1-e^{i\theta})\,-y(1-e^{i\theta})}\,dxdy = \frac{1}{(1-e^{i\theta})^2},$$
$$= \frac{1}{2} - \frac{1}{4}\operatorname{cosec}^2\frac{\theta}{2} - \frac{i}{2}\cot\frac{\theta}{2},$$

if $0 < \theta < 2\pi$.

Equating the real and imaginary parts, we have

$$c = \frac{1}{2} - \frac{1}{4}\operatorname{cosec}^2\frac{\theta}{2}, \text{ and } s = -\frac{1}{2}\cot\frac{\theta}{2},$$

if $0 < \theta < 2\pi$.

(2) Changing θ to $(\pi - \theta)$ in the double cosine series and the double sine series of the previous example, we have

$$\frac{1}{2} - \tfrac{1}{4}\sec^2\frac{\theta}{2} = \quad 1 - \cos\theta + \cos 2\theta - \cos 3\theta + \ldots\ldots\ldots$$
$$- \cos\theta + \cos 2\theta - \cos 3\theta + + \ldots\ldots$$
$$+ \cos 2\theta - \cos 3\theta + \ldots\ldots\ldots\ldots\ldots\ldots$$
$$- \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

provided that $-\pi < \theta < +\pi$;

and
$$-\tfrac{1}{2}\tan\frac{\theta}{2} = \quad 0 + \sin\theta - \sin 2\theta + \sin 3\theta - \ldots\ldots\ldots$$
$$+ \sin\theta - \sin 2\theta + \sin 3\theta - \ldots\ldots\ldots$$
$$- \sin 2\theta + \sin 3\theta - \sin 4\theta + \ldots\ldots$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$- \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

provided that $-\pi < \theta < +\pi$.

(3) Again let $c = \cos\theta + \cos 3\theta + \cos 5\theta + \ldots\ldots\ldots\ldots$
$+ \cos 3\theta + \cos 5\theta + \cos 7\theta + \ldots\ldots\ldots\ldots$
$+ \cos 5\theta + \cos 7\theta + \cos 9\theta + \ldots\ldots\ldots\ldots$
$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

and $s = \sin\theta + \sin 3\theta + \sin 5\theta + \ldots\ldots\ldots\ldots$
$+ \sin 3\theta + \sin 5\theta + \sin 7\theta + \ldots\ldots\ldots\ldots$
$+ \sin 5\theta + \sin 7\theta + \sin 9\theta + \ldots\ldots\ldots\ldots$
$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

As in example (1), it can be easily obtained that

$$c + is = \frac{e^{i\theta}}{(1 - e^{2i\theta})^2}$$

$$= \frac{e^{-i\theta}}{(e^{i\theta} - e^{-i\theta})^2} = \frac{\cos\theta - i\sin\theta}{-4\sin^2\theta}.$$

Equating the real and imaginary parts, we have

$$c = -\frac{\cot\theta \operatorname{cosec}\theta}{4} \text{ and } s = \frac{\operatorname{cosec}\theta}{4},$$

if $0 < \theta < \pi$.

(4) Changing θ into $\frac{\pi}{2} - \theta$, we have

$$\frac{\sec\theta}{4} = \cos\theta - \cos 3\theta + \cos 5\theta - \ldots\ldots$$
$$- \cos 3\theta + \cos 5\theta - \cos 7\theta + \ldots\ldots$$
$$+ \cos 5\theta + \cos 7\theta + \cos 9\theta + \ldots\ldots$$
$$- \ldots\ldots + \ldots\ldots\ldots\ldots\ldots\ldots$$

if $-\frac{\pi}{2} < \theta < +\frac{\pi}{2}$;

and $-\frac{1}{4}\frac{\sec\theta\tan\theta}{4} = \sin\theta - \sin 3\theta + \sin 5\theta + \ldots\ldots$
$- \sin 3\theta + \sin 5\theta - \sin 7\theta + \ldots\ldots$
$+ \sin 5\theta - \sin 7\theta + \sin 9\theta - \ldots\ldots$
$- \ldots\ldots + \ldots\ldots\ldots\ldots\ldots\ldots$

if $-\frac{\pi}{2} < \theta < +\frac{\pi}{2}$.

Some Applications of Modular Equations.

By M. VENKATARAMA AYYAR.

§1. In the paper on "Some Infinite Products and Series" which appeared in the *J.I.M.S.* (Vol. XV, Nos. 7, 10 and 11), certain results relating to 'Theta-Products'* and 'Inverse Tangent Series' of the type $\Sigma_1 - \Sigma_2$† have been established by Mr. Bhimasena Rao and myself. In this paper, I propose to evaluate, by a direct application of a few modular equations, some fresh Theta-Products and also to discuss the sums of the series Σ_1 and Σ_2 separately.

The modular equations made use of in this paper are those of the second, third and fifth orders,‡ *viz*:—

$$\sin\beta = \frac{1-\cos\alpha}{1+\cos\alpha} \quad \ldots \quad \ldots \quad (a)$$

$$(\sin\alpha\,\sin\beta)^{\frac{1}{2}} + (\cos\alpha\,\cos\beta)^{\frac{1}{2}} = 1, \quad \ldots \quad (b)$$

and $$\{(\sin\alpha)^{\frac{1}{2}} - (\sin\beta)^{\frac{1}{2}}\}^6 = 16(\sin\alpha\,\sin\beta)^{\frac{1}{2}}(\cos\alpha\cos\beta)^2 \quad \ldots \quad (c)$$

It may be of interest to note that these modular equations may be derived by simple substitutions in some of the identities given on pages 107 and 93 of Bromwich's *Introduction to Infinite Series*. With Q_0, Q_1, Q_2, Q_3 having the meanings given to them on p. 105 of that book, and with the relations

$$Q_1Q_2Q_3 = 1, \S \quad \ldots \quad \ldots \quad (d)$$

and $$Q_2^8 = Q_3^8 + 16qQ_1^8, \P \quad \ldots \quad \ldots \quad (e)$$

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* *Vide J. I. M. S.*, Vol. XV, No. 7, p. 162, Appendix.

† *Vide J. I. M. S.*, Vol. XV, No. 11, Results VII to XIV.

‡ *Vide* Cayley: *Elliptic Functions*, Art. 244, Art. 261 or 271 (II) and Art. 273 (II).

§ *Vide* Bromwich: *Infinite Series* p. 105 Ex. 14, or Cayley's *Elliptic Functions*, Art. 377.

¶ *Vide* Bromwich; *Infinite Series* p. 107 Ex. 20, or Cayley's *Elliptic Functions*, Art. 377.

putting $\left(\frac{Q_3}{Q_2}\right)^4 = \cos\alpha$, it is easily seen that

$$\left.\begin{aligned} Q_2 &= 2^{\frac{1}{4}} q^{\frac{1}{24}} (\sin 2\alpha)^{-\frac{1}{12}}, \\ Q_3 &= 2^{\frac{1}{4}} q^{\frac{1}{24}} (\sin 2\alpha)^{-\frac{1}{12}} (\cos\alpha)^{\frac{1}{4}}, \\ \text{and} \quad Q_1 &= 2^{-\frac{1}{2}} q^{-\frac{1}{12}} (\sin 2\alpha)^{\frac{1}{6}} (\cos\alpha)^{-\frac{1}{4}}. \end{aligned}\right\} \quad \ldots \text{(A)}$$

Using these relations (A) in the identity given under Ex. 20 on p. 107 of Bromwich, we can, by putting $x = i$ and $x = w$, get, on simplification, the modular equations of the second and third orders.

As a sample of this method, we shall just briefly indicate how the modular equation of the fifth order is arrived at by substitution.

On p. 93 of Bromwich's *Infinite Series*, it has been shown that

$$f(\sqrt{xy}, \sqrt{q}).f\left(\sqrt{\frac{x}{y}}, \sqrt{q}\right) = f(x, q)\, f(y, q) + g(x, q)\, g(y, q)$$

and $f(-\sqrt{xy}, \sqrt{q}).f\left(-\sqrt{\frac{x}{y}}, \sqrt{q}\right) = f(x, q)\, f(y, q) - g(x, q)\, g(y, q).$

where $$f(p, q) = Q_0 . \prod_1^\infty (1 + q^{2n-1} p)\left(1 + \frac{q^{2n-1}}{p}\right)$$

and $$g(p, q) = q^{\frac{1}{6}} p^{-\frac{1}{2}} f\left(\frac{p}{q}, q\right).$$

Multiplying up, substituting for 'f' and 'g,' and simplifying

$$Q_3^4 . \prod_1^\infty \left\{ \left(1 - q^{2n-1} xy\right)\left(1 - \frac{q^{2n-1}}{xy}\right)\left(1 - q^{2n-1}\frac{x}{y}\right)\left(1 - q^{2n-1}\frac{y}{x}\right)\right\}$$

$$= \left[\prod_1^\infty \left\{ \left(1 + q^{2n-1} x\right)\left(1 + \frac{q^{2n-1}}{x}\right)\left(1 + q^{2n-1} y\right)\left(1 + q^{2n-1}\frac{1}{y}\right)\right\}\right]^2$$

$$- q(x^{\frac{1}{2}} + x^{-\frac{1}{2}})^2 (y^{\frac{1}{2}} + y^{-\frac{1}{2}})^2 \times$$

$$\left[\prod_1^\infty \left\{ \left(1 + q^{2n} x\right)\left(1 + \frac{q^{2n}}{x}\right)\left(1 + \frac{q^{2n}}{y}\right)\left(1 + q^{2n} y\right)\right\}\right]^2. \quad \text{(B)}$$

In (B), put $x = w$ (a fifth root of unity) and $y = w^3$. Then, denoting by Q', the infinite product corresponding to Q when q^5 replaces q, we

have

$$Q_3^4 \cdot \left(\frac{Q_3'}{Q_3}\right) = \left(\frac{Q_2'}{Q_2}\right)^2 - q\left(\frac{Q_1'}{Q_1}\right)^2$$

or

$$Q_3 Q_3' = Q_1^2 Q_2'^2 - q Q_2^2 Q_1'^2. \quad \ldots \quad \ldots \quad (C)$$

If β now denote the modular angle for Q', on substituting from (A) and making a few reductions, we get

$$4 \cos\alpha \cos\beta (\sin 2\alpha \sin 2\beta)^{\frac{1}{3}} = \{(\sin\alpha)^{\frac{1}{2}} - (\sin\beta)^{\frac{1}{2}}\}^4. \quad \ldots \quad (1)$$

This is easily seen to be the same as the form in Cayley for the modular equation of the fifth order given in (C). It is also identical with Russel's form

$$\cos(\alpha - \beta) + 2\{\sin 2\alpha \sin 2\beta\}^{\frac{1}{3}} = 1;$$

for, on putting $(\sin 2\alpha \sin 2\beta)^{\frac{1}{3}} = p^2$, (1) becomes

$$(\sqrt{\sin\alpha} - \sqrt{\sin\beta})^2 = 2p \cdot \sqrt{\cos\alpha \cos\beta};$$

which gives

$$\sin\alpha + \sin\beta = 2(p \cdot \sqrt{\cos\alpha \cos\beta} + \sqrt{\sin\alpha \sin\beta}).$$

Squaring and simplifying, we get

$$2p^2 = 1 - \cos(\alpha - \beta)$$

which is Russel's form.

This fifth order modular equation may be put into yet another convenient form which we use below :

$$(\sin 2\alpha \sin 2\beta)^{\frac{1}{3}} = \sin^2\left(\frac{\alpha - \beta}{2}\right).$$

$$\therefore \quad \cos 2(\alpha - \beta) - \cos 2(\alpha + \beta) = 2\sin^6 \frac{\alpha - \beta}{2}.$$

$$\therefore \quad \sin^2(\alpha + \beta) = \sin^2 \frac{\alpha - \beta}{2}\left\{\sin^4 \frac{\alpha - \beta}{2} + 4\cos^2 \frac{\alpha - \beta}{2}\right\}.$$

$$\therefore \quad \sin(\alpha + \beta) = \sin \frac{\alpha - \beta}{2}\left(2 - \sin^2 \frac{\alpha - \beta}{2}\right). \quad \ldots \quad (2)$$

§ 2. It was shown that, when

$$q = e^{-\pi}, \; Q_2 = 2^{\frac{1}{4}} q^{\frac{1}{24}}*,$$

such that, from (A), $(\sin 2\alpha)^{-\frac{1}{12}} = 1$

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* *Vide J. I. M. S.*, Vol. XV, No. 7, p. 162, Appendix.

giving the modular angle $\alpha = 45°$. We shall now find the values of Q_1, Q_2, Q_3 when $e^{-\pi}$ is changed into $e^{-2\pi}$ and $e^{-3\pi}$. In the first case, we have from (*a*),

$$\sin\beta = (\sqrt{2}-1)^2$$

making $$\cos\beta = 2^{\frac{5}{4}}(\sqrt{2}-1),$$

and $$\sin 2\beta = 2^{\frac{3}{4}}(\sqrt{2}-1)^3.$$

$$\text{Hence, } Q_2' = \prod_1^\infty \{1 + e^{-2(2n-1)\pi}\} = 2^{\frac{1}{16}} e^{-\frac{\pi}{12}} (\sqrt{2}-1)^{-\frac{1}{4}} \quad \ldots \quad (3)$$

$$\therefore \quad Q_3' = \prod_1^\infty \{1 - e^{-2(2n-1)\pi}\} = 2^{\frac{3}{8}} e^{-\frac{\pi}{12}} \quad \ldots \quad \ldots \quad (4)$$

$$\text{and} \quad Q_1' = \prod_1^\infty \{1 + e^{-4n\pi}\} = 2^{-\frac{7}{16}} e^{\frac{\pi}{6}} (\sqrt{2}-1)^{\frac{1}{4}} \quad \ldots \quad (5)$$

In the second case, using the modular equation of the third order as in (*b*), *viz.*,

$$(\sin\alpha \sin\beta)^{\frac{1}{2}} + (\cos\alpha \cos\beta)^{\frac{1}{2}} = 1,$$

we have $$\sqrt[4]{2} = (\sin\beta)^{\frac{1}{2}} + (\cos\beta)^{\frac{1}{2}}.$$

Also, $$1 = \sin^2\beta + \cos^2\beta.$$

Solving, $$(\sin\beta)^{\frac{1}{2}} = 2^{-\frac{3}{4}} + 2^{-\frac{5}{4}} 3^{\frac{1}{4}} (\sqrt{3}-1)$$

and $$(\cos\beta)^{\frac{1}{2}} = 2^{-\frac{3}{4}} - 2^{-\frac{5}{4}} 3^{\frac{1}{4}} (\sqrt{3}-1),$$

whence $$\sin 2\beta = 2^{-2}(\sqrt{3}-1)^4.$$

$$\therefore \quad Q_2' = 2^{\frac{1}{4}} e^{-\frac{3\pi}{24}} (\sin 2\beta)^{-\frac{1}{12}}$$

$$= 2^{\frac{1}{4}} e^{-\frac{3\pi}{24}} . 2^{\frac{1}{6}} (\sqrt{3}-1)^{-\frac{1}{3}}.$$

$$\text{Thus} \quad Q_2' = \prod_1^\infty (1 + e^{-3(2n-1)\pi}) = 2^{\frac{5}{12}} e^{-\frac{\pi}{8}} (\sqrt{3}-1)^{-\frac{1}{3}} \quad \ldots \quad (6)$$

$$\therefore \quad Q_3' = \prod_1^\infty (1 - e^{-3(2n-1)\pi}) = 2^{\frac{1}{24}} e^{-\frac{\pi}{8}} (\sqrt{3}-1)^{-\frac{1}{3}} \times$$

$$\{1 - 2^{\frac{1}{2}} 3^{\frac{1}{2}} (\sqrt{3}-1)\}^{\frac{1}{2}} \ldots \quad (7)$$

and $$Q_1' = \prod_1^\infty (1 - e^{-6n\pi}) = 2^{-\frac{11}{24}} e^{\frac{\pi}{4}} (\sqrt{3} - 1)^{\frac{2}{3}} \times$$

$$\{1 - 2^{-\frac{1}{2}} 3^{\frac{1}{4}} (\sqrt{3} - 1)\}^{-\frac{1}{2}} \ldots \quad (8)$$

A similar procedure may be applied to the other results given at the end of Part I of the paper cited at the beginning.

§ 3. We shall now proceed to show how the equations (*a*), (*b*) and (*c*) could be applied to evaluate separately the two parts Σ_1 and Σ_2 which appeared together in the results VII to XII of Part II of the aforesaid paper. Result VII is

$$[\tan^{-1} e^{-\frac{\pi}{2}} \sqrt{\tfrac{13}{3}} - \tan^{-1} e^{-\frac{3\pi}{2}} \sqrt{\tfrac{13}{3}} + \tan^{-1} e^{-\frac{5\pi}{2}} \sqrt{\tfrac{13}{3}} - \ldots.]$$

$$- [\tan^{-1} e^{-\frac{\pi}{2}} \sqrt{13.3} - \tan^{-1} e^{-\frac{3\pi}{2}} \sqrt{13.3} + \tan^{-1} e^{-\frac{5\pi}{2}} \sqrt{13.3} - \ldots.]$$

$$= \frac{\pi}{12} - \tfrac{1}{2} \tan^{-1} \sqrt{\frac{3}{13}}. \quad \ldots \quad (f)$$

Now, it is known that

$$\sum_1^\infty (-1)^{n+1} \tan^{-1} (e^{-(2n-1)\frac{\pi}{2} \cdot \frac{k'}{k}}) = \tfrac{1}{4} \sin^{-1} (k)^*$$

where $k = \sin \alpha$, α being the modular angle.

Hence, if we suppose the first part of the series on the L. H. S., *viz.*,

$$\sum_1^\infty (-1)^{n+1} \tan^{-1} (e^{-(2n-1)\frac{\pi}{2} \cdot \sqrt{\frac{13}{3}}}) = \frac{\alpha}{4},$$

then the 2nd part, *viz.*,

$$\sum_1^\infty (-1)^{n+1} \tan^{-1} (e^{-(2n-1)\frac{\pi}{2} \cdot \sqrt{13.3}}) = \frac{\beta}{4},$$

where α and β are connected by the modular equation of the 3rd order, *viz.*,

$$(\sin \alpha \sin \beta)^{\frac{1}{2}} + (\cos \alpha \cos \beta)^{\frac{1}{2}} = 1.$$

This could be written as

$$(\sin \alpha \sin \beta)^{\frac{1}{2}} = 1 - (\cos \alpha \cos \beta)^{\frac{1}{2}}.$$

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* *Vide* Jacobi: *Fundamenta Nova.*

Squaring and transposing,

$$2(\cos\alpha\cos\beta)^{\frac{1}{2}} = 1 + \cos(\alpha+\beta).$$

Squaring again,

$$4\cos\alpha\cos\beta = 1 + 2\cos(\alpha+\beta) + \cos^2(\alpha+\beta).$$

$$\therefore\ 2\{\cos(\alpha+\beta) + \cos(\alpha-\beta)\} = 1+2\cos(\alpha+\beta)+\cos^2(\alpha+\beta);$$

i.e.,
$$2\cos(\alpha-\beta) - 2 = \cos^2(\alpha+\beta) - 1$$

i.e.,
$$-4\sin^2\frac{\alpha-\beta}{2} = -\sin^2(\alpha+\beta)$$

i.e.,
$$\sin(\alpha+\beta) = 2\sin\frac{\alpha-\beta}{2}\dagger \quad \ldots \quad \ldots \quad \text{(D)}$$

convenient form used below.

We have, from (*f*),

$$\frac{\alpha}{4} - \frac{\beta}{4} = \frac{\pi}{12} - \tfrac{1}{2}\tan^{-1}\sqrt{\frac{3}{13}}$$

$$= \frac{\pi}{12} - \tfrac{1}{2}\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$$

$$\therefore\ \frac{\alpha-\beta}{2} = \frac{\pi}{6} - \sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$$

$$= \sin^{-1}\left(\frac{\sqrt{13}-3}{8}\right).$$

But, from (D),

$$\alpha+\beta = \sin^{-1}\left\{2\sin\frac{\alpha-\beta}{2}\right\}$$

$$= \sin^{-1}\left(\frac{\sqrt{13}-3}{4}\right).$$

$$\therefore\ \frac{\alpha+\beta}{2} = \tfrac{1}{2}\sin^{-1}\left(\frac{\sqrt{13}-3}{4}\right).$$

Using this with

$$\frac{\alpha-\beta}{2} = \sin^{-1}\left(\frac{\sqrt{13}-3}{8}\right),$$

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† This form is mentioned by the late S. Ramanujan, F. R. S., *vide J. I. M. S.* Vol. IV, p, 78.

we have,

$$\alpha = \tfrac{1}{2}\sin^{-1}\left(\frac{\sqrt{13}-3}{4}\right) + \sin^{-1}\left(\frac{\sqrt{13}-3}{8}\right)$$

and

$$\beta = \tfrac{1}{2}\sin^{-1}\left(\frac{\sqrt{13}-3}{4}\right) - \sin^{-1}\left(\frac{\sqrt{13}-3}{8}\right).$$

Hence,

$$\sum_1^\infty (-1)^{n+1}\tan^{-1}\left(e^{-(2n-1)\frac{\pi}{2}\cdot\frac{\sqrt{13}}{3}}\right).$$

$$= \tfrac{1}{8}\sin^{-1}\left(\frac{\sqrt{13}-3}{4}\right) + \tfrac{1}{4}\sin^{-1}\left(\frac{\sqrt{13}-3}{8}\right). \quad (9)$$

This is a solution of the second part of Question 684 in the *J. I. M. S.**

Also,
$$\sum_1^\infty (-1)^{n+1}\tan^{-1}\left(e^{(2n-1)\frac{\pi}{2}\cdot\sqrt{13.3}}\right)$$

$$= \tfrac{1}{8}\sin^{-1}\left(\frac{\sqrt{13}-3}{4}\right) - \tfrac{1}{4}\sin^{-1}\left(\frac{\sqrt{13}-3}{8}\right). \quad (10)$$

Again, Result XXV of the paper cited above gives

$$\sum_1^\infty (-1)^{n+1}\tan^{-1}\left(e^{-(2n-1)\frac{\pi}{2}\cdot\frac{k'}{k}}\right) + \sum_1^\infty (-1)^{n+1}\tan^{-1}\left(e^{-(2n-1)\frac{\pi}{2}\cdot\frac{k}{k'}}\right)$$

$$= \frac{\pi}{8}.$$

$$\therefore \quad \sum_1^\infty (-1)^{n+1}\tan^{-1}\left(e^{-(2n-1)\frac{\pi}{2}\cdot\sqrt{\frac{3}{13}}}\right)$$

$$= \frac{\pi}{8} - \tfrac{1}{8}\sin^{-1}\left(\frac{\sqrt{13}-3}{4}\right) - \tfrac{1}{4}\sin^{-1}\left(\frac{\sqrt{13}-3}{8}\right). \quad \ldots (11)$$

To results IX and X of the same paper, we could similarly apply the modular equation of the 3rd order, by utilising result VI. But this work has already been done by Mr. M. Bhimasena Rao in his paper on "The Modular Equation of the Third Order."†

Next, taking result XI of the joint paper,

$$\{\tan^{-1} e^{-\frac{\pi}{2}\sqrt{\frac{5}{11}}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{\frac{5}{11}}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{\frac{5}{11}}} - \ldots\ldots\}$$

$$-\{\tan^{-1} e^{-\frac{\pi}{2}\sqrt{5.11}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{5.11}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{5.11}} - \ldots.\}$$

$$= \frac{\pi}{4} - \tfrac{1}{2}\tan^{-1}\sqrt{\frac{11}{5}}.$$

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* *Vide*: *J. I. M. S.*, Vol. VII, No. 4.

† *Vide*: *J. I. M. S.*, Vol. XV, No. 6, results on p. 135,

Now, let

$$\tan^{-1} e^{-\frac{\pi}{2}\sqrt{\frac{11}{5}}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{\frac{11}{5}}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{\frac{11}{5}}} - \ldots = \tfrac{1}{4}\alpha. \qquad (G)$$

Then, from result XXV of the joint paper

$$\tan^{-1} e^{-\frac{\pi}{2}\sqrt{\frac{5}{11}}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{\frac{5}{11}}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{\frac{5}{11}}} - \ldots = \frac{\pi}{8} - \frac{1}{4}\alpha; \quad (H)$$

and also

$$\tan^{-1} e^{-\frac{\pi}{2}\sqrt{5.11}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{5.11}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{5.11}} - \ldots = \frac{1}{4}\beta,$$

where β and α are connected by the modular equation of the fifth order of which we may use the form (2), *viz.*,

$$\sin(\alpha + \beta) = 2\sin\frac{\alpha - \beta}{2} - \sin^3\frac{\alpha - \beta}{2}.$$

If $\dfrac{\alpha - \beta}{2} = \sin^{-1} p$, then $\alpha + \beta = \sin^{-1}(2p - p^3)$.

But, we have $$\frac{\pi}{8} - \frac{1}{4}\alpha - \frac{1}{4}\beta = \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\sqrt{\frac{11}{5}}.$$

$$\therefore \qquad \alpha + \beta = \sin^{-1}(\tfrac{3}{8}).$$

$$\therefore \qquad 2p - p^3 = \tfrac{3}{8}.$$

This gives an admissible solution $p = \dfrac{3 - \sqrt{5}}{4}$.

$$\therefore \qquad \alpha + \beta = \sin^{-1}\tfrac{3}{8}; \; \alpha - \beta = 2\sin^{-1}\frac{3 - \sqrt{5}}{4}$$

or $$\frac{\alpha}{2} + \frac{\beta}{2} = \sin^{-1}\frac{\sqrt{11} - \sqrt{5}}{4\sqrt{2}},$$

and $$\frac{\alpha}{2} - \frac{\beta}{2} = \sin^{-1}\frac{3 - \sqrt{5}}{4};$$

such that

$$\alpha = \sin^{-1}\frac{\sqrt{11} - \sqrt{5}}{4\sqrt{2}} + \sin^{-1}\frac{3 - \sqrt{5}}{4},$$

and $$\beta = \sin^{-1}\frac{\sqrt{11} - \sqrt{5}}{4\sqrt{2}} - \sin^{-1}\frac{3 - \sqrt{5}}{4}.$$

Hence,

$$\tan^{-1} e^{-\frac{\pi}{2}\sqrt{\frac{11}{5}}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{\frac{11}{5}}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{\frac{11}{5}}} - \ldots\ldots$$

$$= \tfrac{1}{4}\left\{\sin^{-1}\frac{\sqrt{11}-\sqrt{5}}{4\sqrt{2}} + \sin^{-1}\frac{3-\sqrt{5}}{4}\right\}, \quad \ldots \quad (12)$$

$$\tan^{-1} e^{-\frac{\pi}{2}\sqrt{5.11}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{5.11}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{5.11}} - \ldots\ldots$$

$$= \tfrac{1}{4}\left\{\sin^{-1}\frac{\sqrt{11}-\sqrt{5}}{4\sqrt{2}} - \sin^{-1}\frac{3-\sqrt{5}}{4}\right\}, \quad \ldots \quad (13)$$

and

$$\tan^{-1} e^{-\frac{\pi}{2}\sqrt{\frac{5}{11}}} - \tan^{-1} e^{-\frac{3\pi}{2}\sqrt{\frac{5}{11}}} + \tan^{-1} e^{-\frac{5\pi}{2}\sqrt{\frac{5}{11}}} - \ldots\ldots$$

$$= \tfrac{1}{4}\left\{\frac{\pi}{2} - \sin^{-1}\frac{\sqrt{11}-\sqrt{5}}{4\sqrt{2}} - \sin^{-1}\frac{3-\sqrt{5}}{4}\right\}. \quad \ldots \quad (14)$$

In conclusion, I desire to acknowledge my indebtedness to Mr. M. Bhimasena Rao at whose suggestion this paper was written.

On the relation between the Convergence of a Series and its Summability by Cesaro's Means*

By K. ANANDA RAU.

1. In the first of a series of important papers communicated to the London Mathematical Society, Hardy proved the following :

THEOREM I.† *If* Σa_n *is summable* (C 1) *to sum* s, *and if*

$$a_n = O\left(\frac{1}{n}\right), \tag{1}$$

then Σa_n *converges to sum* s.

In case when the a's are real, Landau‡ made an extension of this theorem by replacing condition (1) by

$$a_n > -\frac{K}{n},$$

where K is a positive constant independent of n. The object of this paper is to make a further extension of the theorem in a somewhat different direction. The generalization consists in this, that we replace the sequence $\left(\frac{K}{n}\right)$ by a general sequence (b_n)§ and postulate on the asymptotic behaviour of the sum

$$b_1 + b_2 + \ldots\ldots + b_n. \tag{2}$$

The proof of the generalized theorem is obtained by modifying that of Theorem I and its "left-handed" extension. It may make the genesis of the generalization clear, if I point out the relation between the restriction that is to be imposed on the sum (2) and the known properties of the sequence $\left(\frac{K}{n}\right)$.

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* A short abstract of this papar was printed in the Records of the Proceedings of the London Mathematical Society, Vol. 18 (1920), p. xxii.

† "Theorems relating to the Summability and Convergence of Slowly Oscillating Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 301—320.

‡ *Prac Matematyczno-Fizycznych*, Vol 21 (1910), pp. 97—177.

§ The terms of this sequence are not necessarily positive.

We know that

$$\frac{K}{1}+\frac{K}{2}+\ldots+\frac{K}{n}=K\log n+K'+o(1)$$

where
$$\psi(n)\equiv K\log n+K'$$

satisfies the condition

$$\lim_{\delta\to 0}\ \overline{\lim_{n\to\infty}}\ [\psi(n+n\delta)-\psi(n)]=0.$$

The restriction that we impose on the sum (2) is that, as $n\to\infty$,

$$\tau_n=\sum_1^n b_n=\chi(n)+o(1)$$

where $\chi(n)$ is a positive increasing function such that

$$\lim_{\delta\to 0}\ \overline{\lim_{n\to\infty}}\ [\chi(n+n\delta)-\chi(n)]=0 \tag{3}$$

2. The theorem runs as follows :

THEOREM II. *Suppose that the sequence* (b_n) *satisfies the condition stated above. Let* Σa_n *be summable* $(C\,1)$ *to sum* s, *and let*

$$a_n+b_n\geqslant 0$$

for sufficiently large values of n. *Then* Σa_n *converges to sum* s.

Let
$$s_n=a_1+a_2+\ldots\ldots+a_n$$
$$\sigma_n=\frac{s_1+s_2+\ldots\ldots+s_n}{n}.$$

We may clearly suppose that $s=0$. If the theorem were false, there is an $h>0$, such that one at least of the inequalities

(i) $s_n>h$

(ii) $s_n<-h$

is satisfied for an infinity of values of n. We shall prove that either of these suppositions leads to a contradiction. Let us consider, for instance, the supposition (i).

First choose a positive $\delta<1$ so that

$$\phi(\delta)=\lim_{n\to\infty}[\chi(n+n\delta)-\chi(n)]<\frac{h}{48}.$$

Keeping δ fixed, choose N_1 so that for $n > N_1$

$$\chi(n + n\delta) - \chi(n) < \phi(\delta) + \frac{h}{48};$$

we shall then have for $n > N_1$

$$\chi(n + n\delta) - \chi(n) < \frac{h}{24}. \tag{4}$$

Next choose N_2 so that for $n > N_2$

$$|\sigma_n| < \frac{1}{8}\frac{h\delta}{1+\delta}, \tag{5}$$

$$|\tau_n - \chi(n)| < \frac{h}{24}, \tag{6}$$

$$a_n + b_n \geqslant 0.$$

Let n be such that

$$n > \max(N_1, N_2).$$

$$s_n > h.$$

Let $r = r(n)$ be the greatest integer in $n\delta$.

Now, for $p = 1, 2, \ldots\ldots r$,

$$\begin{aligned} s_{n+p} &= s_n + a_{n+1} + \ldots\ldots + a_{n+p} \\ &> h - (b_{n+1} + \ldots\ldots + b_{n+p}) \\ &= h - (\tau_{n+p} - \tau_n). \end{aligned} \tag{7}$$

$$\begin{aligned} \sigma_{n+r} &= \frac{s_1 + s_2 + \ldots\ldots + s_n + \ldots\ldots + s_{n+r}}{n+r} \\ &= \frac{s_1 + s_2 + \ldots\ldots + s_n}{n}\,\frac{n}{n+r} + \frac{s_{n+1} + \ldots\ldots + s_{n+r}}{n+r} \end{aligned}$$

Hence by (5) and (7),

$$\sigma_{n+r} > -\frac{1}{8}\frac{h\delta}{1+\delta} + \frac{hr}{n+r} - \frac{1}{n+r}\sum_{p=1}^{r}(\tau_{n+p} - \tau_n) \tag{8}$$

Now

$$\frac{hr}{n+r} \geqslant \frac{1}{2}\frac{h(r+1)}{n+r} > \frac{h}{2}\,\frac{\delta}{1+\delta}, \tag{9}$$

since

$$r + 1 > n\delta,\ r \leqslant n\delta.$$

Further,

$$(10)\qquad \left|\sum_{p=1}^{r}(\tau_{n+p}-\tau_n)\right| \leqslant \sum_{p=1}^{r}\left|\tau_{n+p}-\chi(n+p)\right| + \sum_{p=1}^{r}\left|\tau_n-\chi(n)\right| + \sum_{p=1}^{r}\left|\chi(n+p)-\chi(n)\right|.$$

The first two sums on the right are each less than $\frac{hr}{24}$ by (6). As for the last term, we have

$$\sum_{p=1}^{r}\left|\chi(n+p)-\chi(n)\right| \leqslant r\,[\chi(n+r)-\chi(n)] \leqslant r\,[\chi(n+n\delta)-\chi(n)] < \frac{hr}{24}$$

by (4). Hence

$$(11)\qquad \frac{1}{n+r}\left|\sum_{p=1}^{r}(\tau_{n+p}-\tau_n)\right| < \frac{hr}{8(n+r)} < \tfrac{1}{8}\,\frac{h\delta}{1+\delta}.$$

Using (9), (10) and (11) in (8), we get

$$\sigma_{n+r} > -\tfrac{1}{8}\frac{h\delta}{1+\delta} + \tfrac{1}{2}\frac{h\delta}{1+\delta} - \tfrac{1}{8}\frac{h\delta}{1+\delta}$$

$$> \tfrac{1}{4}\frac{h\delta}{1+\delta}.$$

Clearly $(n+r) \to \infty$ as $n \to \infty$. Hence for an infinity of values of m, we have

$$\sigma_m > \frac{h\delta}{4(1+\delta)},$$

which is contradictory to the hypothesis that $\sigma_m \to 0$.

We may similarly dispose of the supposition (ii). In this case, we consider σ_{n-r} instead of σ_{n+r} and remembering that $\delta < 1$, $(n-r) \to \infty$, we prove that for an infinity of values of m

$$\sigma_m < -\frac{h\delta}{4(1+\delta)},$$

which is also contradictory to the hypothesis of the theorem.

The proof is therefore complete.

3. If for a given series Σa_n, we take

$$b_n = |a_n|$$

then the condition $$a_n + b_n \geqslant 0$$

is clearly satisfied. Also, if we write

$$\chi(n) = \sum_1^n |a_n|,$$

then $\chi(n)$ is a positive increasing function and the condition (3) becomes

$$\lim_{\delta \to 0} \overline{\lim_{n \to \infty}} \sum_n^{n(1+\delta)} |a_n| = 0.$$

Theorem II therefore reduces to the following :

THEOREM III. *Suppose that* Σa_n *is summable* $(C\,1)$ *and that*

$$\lim_{\delta \to 0} \overline{\lim_{n \to \infty}} \sum_n^{n(1+\delta)} |a_n| = 0;$$

then Σa_n *is convergent.*

The Group Concept

By G. A. MILLER.

1. The word *group* seems to have originated in Central Europe and to be etymologically akin to the word 'crop.' A mathematical group may profitably be regarded as a mathematical crop for which seeds were being scattered very early in the development of our subject but which received little attention and was not thoughtfully cultivated until recent times. Under such cultivation it has yielded rich harvests whose importance has been increasingly recognized, especially during the last fifty years, and there seems to be no evidence which tends to show that we have as yet fully realised its value and its varied usefulness. New and important applications are being discovered in rapid succession and only few mathematicians are as yet in a position to evaluate their full significance.

The word group was in use for a long time before E. Galois began to use it as a technical mathematical term in about 1830. He used it also as a non-technical term in the sense of a collection, and later mathematical writers have followed him in this respect. For instance, we still speak of a group of points, a group of waves, group insurance, etc., as well as of a group of substitutions, a group of movements, a group of operations, etc. What is of still greater importance to the beginner in this field of study is the fact that, even in the best modern mathematical literature, the technical term group is being used with widely different meanings as may be seen from the different definitions of this term in the *Encyclopédie des Sciences Mathematiques*, tome 1, volume 1, page 576; and tome 1, volume 2, page 243.

2. The most general technical mathematical meaning of the word group may be obtained by restricting its non-technical meaning: that is, by imposing the condition that each of the elements of the collection can be combined with itself and also with every other element of the collection. Moreover, the result obtained by such a combination must always be an element of the collection. This closure property is involved in every

mathematical definition of the term group, and it was the only one imposed on the elements under consideration by the early writers. This was due to the fact that these elements were concrete objects which by their nature satisfy the other conditions which are now usually imposed on a set of abstract elements which constitute a group.

It should be noted that the concept of group appeared in the mathematical literature in an implicit form more than three thousand years before any special name was assigned to this concept in its modern abstract form. In a special concrete form the concept has had various names. The earliest of these names was doubtless the name for unity. In fact, the positive rational numbers, exclusive of zero, constitute a group in the modern restricted sense of the term with respect to the operation known as multiplication, and all the number systems of the various civilizations existing before the Christian era were included in this group of numbers, which may be regarded as the most important mathematical group. From the work of Ahmes, written about 1700 B.C., it is clear that the ancient Egyptians had acquired considerable insight into this group and some of its sub-groups. Their word for number may possibly have represented this group.

The sub-group of the positive rational number group, which is formed by the powers of 10, played a very fundamental role in the development of the various number systems. Among the Babylonians and the ancient Greeks, the sub-group formed by the powers of 60 received especial attention, and this sub-group receives still some attention in our measurement of time and angles. It would doubtless be claiming too much to say that the ancient Egyptians and the ancient Babylonians had a clear notion of the group formed by the positive rational numbers. In fact, even the ancient Greeks failed to be sufficiently influenced by the properties of this group and its various sub-groups, otherwise they would have introduced negative numbers. Such numbers are essential not only for the study of the sub-groups—in the modern restricted sense of the term group—formed by certain arithmetical series, but also of those formed by certain geometrical series, including those formed by the powers of 10 and the powers of 60.

Groups have been divided into *four* categories, as follows: finite discontinuous, infinite discontinuous, finite continuous and infinite continuous. All the groups and sub-groups noted above, with the exception of the group formed by unity, belong to the second of these categories. Notwithstanding the fact that, with the exception of trivial cases, the earliest groups which played an important part in the development of mathematics were of infinite order yet it was the finite discontinuous groups which first led to a general theory of groups. These groups began to be studied during the latter half of the eighteenth century in connection with the solution of algebraic equations in one unknown, and the terminology thus developed was largely transferred to similar concepts arising in connection with the study of the other categories. Among the important concepts common to all of these categories are the following : sub-group, invariant sub-group, conjugate groups and quotient groups.

3. We referred above to the most general technical definition of the term group. The most common additional restriction imposed on the elements of a group is that each element of the collection can be combined with at least one element of the collection so as to obtain the identity. In other words, each of the elements of the group shall have at least one inverse. It should be noted that in Lie's theory of transformation groups this restriction is not always imposed. In what follows we shall assume that when the elements of a group are combined the associative law is satisfied, and if any two of the symbols of the equation

$$xy = z$$

are replaced by elements of the group the equation has always one and only one root in the group, which is assumed to be composed of distinct elements. This equational definition of the term group appears to the writer to be the most satisfactory of all those which have been advanced.

4. A central feature of the development of group theory has been the establishment of an $(\infty, 1)$ correspondence. While there is an infinite number of algebraic equations of degree n there is only a finite number of groups on n letters, and many of the fundamental properties of these equations are reflected in the properties of the corresponding groups.

The theory of the possible substitution groups on n letters may appear to involve many details; but when it is noted that each of these possible groups reflects light on an infinite system of equations these details assume a greater significance. In particular, during recent years a theory relating to the cycles of the substitutions involved in a group has received some attention. It is obvious that such a theory can only be expected to attract general interest on account of the $(\infty, 1)$ correspondence between substitution groups and various other fields of mathematical interest.

The number of elements involved in a finite discontinuous group is said to be its *order* and the existence of at least one group of any given finite order n is established by the fact that the n rotations through an angle $2\pi/n$ and its multiples obviously constitute a group of order n. We thus have a very simple existence theorem relating to the group concept, and the establishment of existence theorems as regards other possible groups has constituted a very fruitful field of investigation. While these investigations have led to many new and unsuspected relationships they have not encouraged those who would like to think that the intellectual space of human endeavour is finite. On the contrary, new difficulties present themselves much more rapidly here than the overcoming of old ones. The growing horizon of intellectual penetration becomes also a growing horizon of intellectual difficulties. In particular, there is as yet no basis in sight for the hope that we shall ever know all the simple discontinuous groups of finite order.

5. While existence theorems as regards possible groups have played a fundamental role in the development of the group concepts, a still more fundamental role has been played by the wide applications of this concept to various branches of mathematics. It is probable that H. Poincare had such applications in mind when he said towards the close of his life: "The theory of groups is, so to say, entire mathematics divested of its matter and reduced to pure form."* Such applications seem also to have been in the mind of E. Study when he said, :" The word elegance denotes only to some extent what is involved here. It is

* *Acta Mathematica*, Vol. 38 (1921), p. 145.

a colossal misunderstanding to regard as only outside matters that which is at stake as regards the innermost character of mathematics. It is a question of conformity to laws which are mostly of a group theoretic character." *

In a recent number of *Scientia*, Volume 18 (1924), page 145, G. Fano published an article entitled: "*I gruppi di transformazione nella geometria*" in which he points out the fundamental position of the group concept in the development of geometry. In particular, he again calls attention to the fact that the best answer to the question 'what is a geometric property?' is one which starts from the concept of group of transformations (or of operations), a fundamental concept which during a century is on the way of becoming very important both as a source of information and as a principle for classifying the ancient and recent theories. Later in the same article, he asserts that the concept of group of operations is one of those which govern all modern mathematics. The present writer has often been surprised to find so many other very emphatic statements relating to the importance of the concept of groups, and hence he published a brief list of such statements in the *Tohoku Mathematical Journal*, Vol. 6 (1914), page 73; a somewhat more extended list was later published in *Revista Matematica Hispano-Americano*, Volume 1 (1919), page 148. It would probably be impossible to provide an equally imposing list of favourable comments on the part of leading mathematicians relating to any other modern mathematical concept.

6. An essential element of the concept of group is that it deals with a totality of operations. The simplest such totality consists of the identity operation and the group constituted by it is sometimes called the *principal group* (*Hauptgruppe*). F. Klein used the same term for the continuous mixed group of movements, similitudes and symmetries—the study of whose invariants constitutes elementary geometry. In a certain sense the identity operation is no operation at all and it is not known who first regarded it as an operation. Since every group must involve the identity operation, it is obvious that *the concept of group with its modern restricted*

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* E. Study: *Einleitung in die Theorie der Invarianten linearer Transformationen auf Grund der Vektorenrechnung* 1923, Introduction.

meaning could not have existed prior to the concept of the identity operation; and conversely. In the group formed by the positive rational numbers when they are combined by multiplication, the number 1 constitutes the identity operation and this fact must have been observed very early in the development of the number concept.

7. When we stated above that the oldest name of a group was the name for unity it was not implied that those who first used this name were impresed by the fact that the number 1 is a group as regards the operation of multiplication. It may be that the group property of the number 1 as regards the operation of multiplication was not observed any earlier than that of the positive rational numbers when they are combined according to the same operation. In the former case, this property appears at first to be more trivial than in the latter, and hence it is conceivable that these two groups were discovered at the same time, but the former doubtless received a name long before it was regarded as a group. As regards the latter, it is not so clear that the name number was assigned to it before its group property with respect to multiplication was observed. Mathematicians have probably always differed widely as regards keen distinctions. It is of interest to know when people first began to use groups even though it is impossible to determine whether they were aware of their existence, or whether they were even vaguely guided thereby.

8. The growing popularity, rather than the growing appreciation, of the group concept is reflected in the number of books explicitly devoted thereto. The first of these was C. Jordan's *Traite des substitutions et des equations algebraiques*, 1870. This was the first book published by its author (with the exception of his thesis), and represents a field in which he was interested until the end of his very fruitful mathematical career. It contains much that was new and it inspired much of the later work along the same line; but it was never republished in French or any other language. About 12 years after its publication, Netto's *Substitutionentheorie* (1882) appeared. This was a more elementary work. An Italian edition of it, prepared by G. Battaglini, appeared in 1885, and an English edition pre-

pared by F. N. Cole appeared in 1892. Both of these works were devoted to finite discontinuous groups; but the next two large works which were explicitly devoted to the theory of groups were devoted to finite continuous groups. The former of these appeared in three volumes, entitled *Theorie der Transformationsgruppen* (1888—1893), by S. Lie and F. Engel, while the latter appeared in a single volume under the title *Vorlesungen uber kontinuierliche Gruppen*, 1893, by S. Lie and G. Scheffers.

After the publication of these four works other books devoted explicitly to the group concept began to appear more rapidly; but the number of these books is still small, since the number of students who have made an extensive study of this concept has always been relatively small. Among the books which aim to treat a relatively large portion of the subject are the following:—*Theory of Groups of Finite Order*, 1897, by W. Burnside, second edition, 1911; *Nozioni sulle transformazioni puntuali e sui gruppi continui*, 1895, by F. Giudice; *Teoria dei gruppi di transformazioni*, 1898, by G. Vivanti, translated into French by A. Boulanger, 1904; *Lezioni sulla teoria dei gruppi di sostituzioni*, 1900, by L. Bianchi; *Linear groups*, 1901, by L. E. Dickson; *Theory of continuous groups*, 1903, by J. E. Campbell; *Groupes abstraits*, 1904, by J. A. de Seguier; *Teoria dei gruppi discontinui e delle funzioni automorfe*, 1908, by G. Fubini; *Finite groups*, 1908, by H. Hilton; *Gruppen-und substitutionentheorie*, 1908, by E. Netto: *Groupes de substitutions*, 1912, by J. A. de Seguier; *Theory and applications of finite groups*, 1916, by Miller, Blichfeldt and Dickson; *Lezioni sulla teoria dei gruppi continui finiti di transformazioni*, 1918, by L. Bianchi; *Theorie der Gruppen von endlicher Ordnung*, 1923, by A. Speiser.

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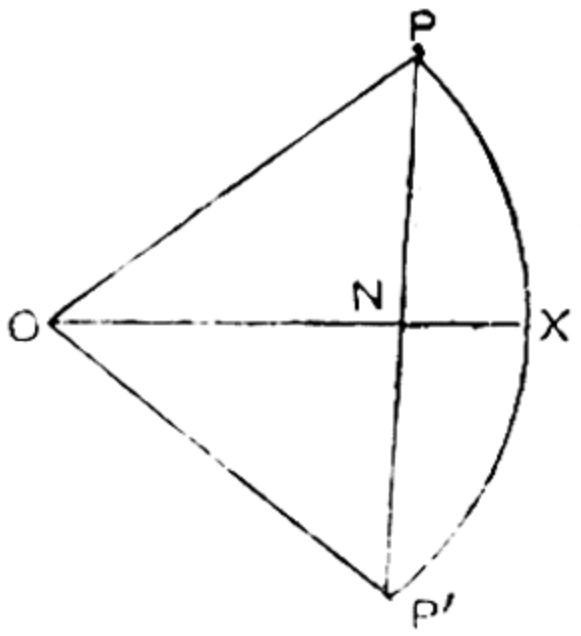
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Notes and Questions.

Note on the Hindu Table of Sines.

The early notions of arc and sine are derived from archery where the full arc PXP′ resembles the bow (L. *arcus*, bow, चाप) and the full chord PNP′ the bow-string (L. *sinus*, bosom, शिंजिनी, जीव or ज्या). But in mathematics, it is only the upper halves PX and PN that admittedly have any purpose to serve[1]; and though our old writers first employ the



term ज्यार्ध, sine-half, etc., in order to fix the nature of the line on students' minds[2], they soon drop the qualifying word अर्ध half[3].

The old conventions did not, like the modern ones, treat sine as a fraction and radius as unity, but gave certain values to the

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[1] [अत्र गणिते ऊर्ध्वरेखातोऽर्धज्याया एव प्रयोजनात्—in this mathematical science it is only the upper half that has any purpose to serve (Ranganatha); and अर्ध ज्याभिः कर्म सर्वं ग्रहाणामर्ध ज्यैव ज्याभिधानात्त वेद्या --all calculations relating to the planets are made by taking half-sine, and therefore let it be understood that half sine (ज्यार्ध) is itself treated as sine (ज्या) in this science (Bhaskara-charya)]

[2] (पूर्वं तु स्वरुपो कत्यर्थं पिंडानां ज्यार्धे त्योक्तिः,—previously however their values were spoken of as sine-halves for showing their nature.

[3] इदानीं तेषामेवार्धत्यागेन ज्या पिंडत्वोक्तिः,—but now the very same values are spoken of as ज्या's, sines, without the word अर्ध half.

lines themselves. Further, the Hindus in their zeal to avoid fractions and decimals in their calculations gave sufficiently large values to the lines themselves so as to keep the results within the range of accuracy. They regarded the circumference of a circle as containing 360° or 21600′; and the ratio of the circumference of a circle to its diameter, π, was given by

भनन्दाग्नि मितपरिधौ, ख बाण सूर्यमितो व्यासः[4]

i.e. when the circumference measures 3927, the diameter measures 1250. (Bhãskarâchârya, cited with approval by Ranganâthâ). Thus

$$\frac{\text{circumference of the circle}}{\text{diameter of the circle}} = \pi = \frac{3927}{1250}$$

$$= \frac{3927 \times 8}{1250 \times 8} = \frac{31416}{10000}$$

$$= 3{\cdot}1416.$$

the modern figure itself correct to four places of decimals; the radius (त्रिज्या, so called because it is त्रिराशिज्या, or the sine of $3 \times 30°$ $= 90°$) was treated as 3438′ (a radian).

To construct a table of 24 sines they divided the arc of a quadrant into 24 parts of 225′ each. Again, this choice of 24 is based upon a very important observation, familiar even from very early days as is borne out by a statement of Sakalya[5],

वृत्तस्य षण्णवत्यंशो दंडवद्दृश्यते तु सः ।

that is, "one-ninety-sixth part of a circle looks (straight) like a rod." Chamber's *Mathematical Tables* give

$$\sin\ 3° 45′ = {\cdot}0654031 = {\cdot}06540;$$
$$\text{arc}\ 3° 45′ = {\cdot}0654498 = {\cdot}06545;$$
$$\text{and} \quad \tan 3° 45′ = {\cdot}0655435 = {\cdot}06554,$$

correct to 5 places of decimals. Thus for an angle of 3° 45′ or 225′, the arc is in excess of the sine by 5, and the tangent is in excess of the arc by 9 in 100,000 and these differences may be considered as inappreciable. Certainly a set of Tables calculated at shorter intervals than 225′ would give more exact values and the Hindus who divided 1″ (vikalâ) into 60 paras, 1 para into 60 paratparas,

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[4] भ = 27, नंद = 9, अग्नि = 3, ख = 0, बाण = 5, सूर्य = 12.

[5] Apparently the son of Sakala who first taught Rigveda in its present form.

and 1 paratpara into 60 tatparas, and also marked the division of time into as a unit as त्रुटि ($= \frac{1}{33750}$ of a second) were certainly competent to realize this and employ a smaller unit, if they would, for the purpose. But then it would have been consistent to construct a longer table involving more figures too. Apparently this did not appeal to them as they had to trust to their memory as their ready guide, and they chose the present middle course and left the individual to make his own calculations employing as small a unit as he would.

The following rule for calculating the values of sines is given in the Surya-Siddhanta:—

राशिलिप्ताष्टमो भागः प्रथमं ज्यार्धमुच्यते ।
तत्तद्विभक्तलब्धोन मिश्रितं तद्द्वितीयकं ॥
आद्ये नैवं क्रमात्पिंडान् भक्त्वा लब्धोन संयुताः ।
खण्डकाः स्युश्चतुर्विंशज्ज्यार्ध पिंडाः क्रमादमी ॥

(II, 15—16)

This passage has to be carefully followed as it is likely to mislead; its purport, gathered in the light of the commentary, could be expressed thus: (1) one-eighth part of *rasi* (*i. e.*, $\frac{1}{8}$ of $30^\circ = 3^\circ 45' = 225'$) is the first sine; (2) divide it by itself, subtract the *quotient* from itself, add the remainder to itself, and the second sine is obtained; (now, the rule generalizes the process by induction, which is, practically, though not identically, the modern mathematical induction); (3) divide the successive sines already obtained by 225, subtract *the sum of all such quotients* from 225, add the remainder to the last of the sines already obtained, and the result gives the next sine.

In other words, let

S_r denote the r^{th} sine,

and Q_r ,, the r^{th} quotient in the successive divisions stated above;

then the rule says,

$$S_1 = 225\ ;$$
$$S_2 = S_1 + (225 - Q_1), \text{ etc.}$$

and generally

$$S_r = S_{r-1} + (225 - \overline{Q_1 + Q_2 + Q_3 + \ldots\ldots + Q_{r-1}})$$
$$= S_{r-1} + S_1 - \Sigma Q_{r-1}$$

$$\therefore \qquad S_r - S_{r-1} = S_1 - \Sigma Q_{r-1},$$

which gives the excess of one sine over the one immediately preceding it : *i.e.*, if D_{r-1} denote this excess or the difference to be added to the $(r-1)^{th}$ sine to give the r^{th} sine, then

$$D_{r-1} = S_1 - \Sigma Q_{r-1},$$

and similarly $$D_r = S_1 - \Sigma Q_r ;$$

$$\therefore \qquad D_{r-1} - D_r = \Sigma Q_r - \Sigma Q_{r-1}$$

i. e., $$D_{r-1} - D_r = Q_r.$$

Hence each successive difference is in defect of the difference immediately preceding it by Q_r.

Now giving different values to r,

$$Q_1 = \frac{S_1}{225} = \frac{225}{225} = 1 ; \therefore S_2 = S_1 + (225 - 1)$$
$$= 225 + 224 = 449.$$

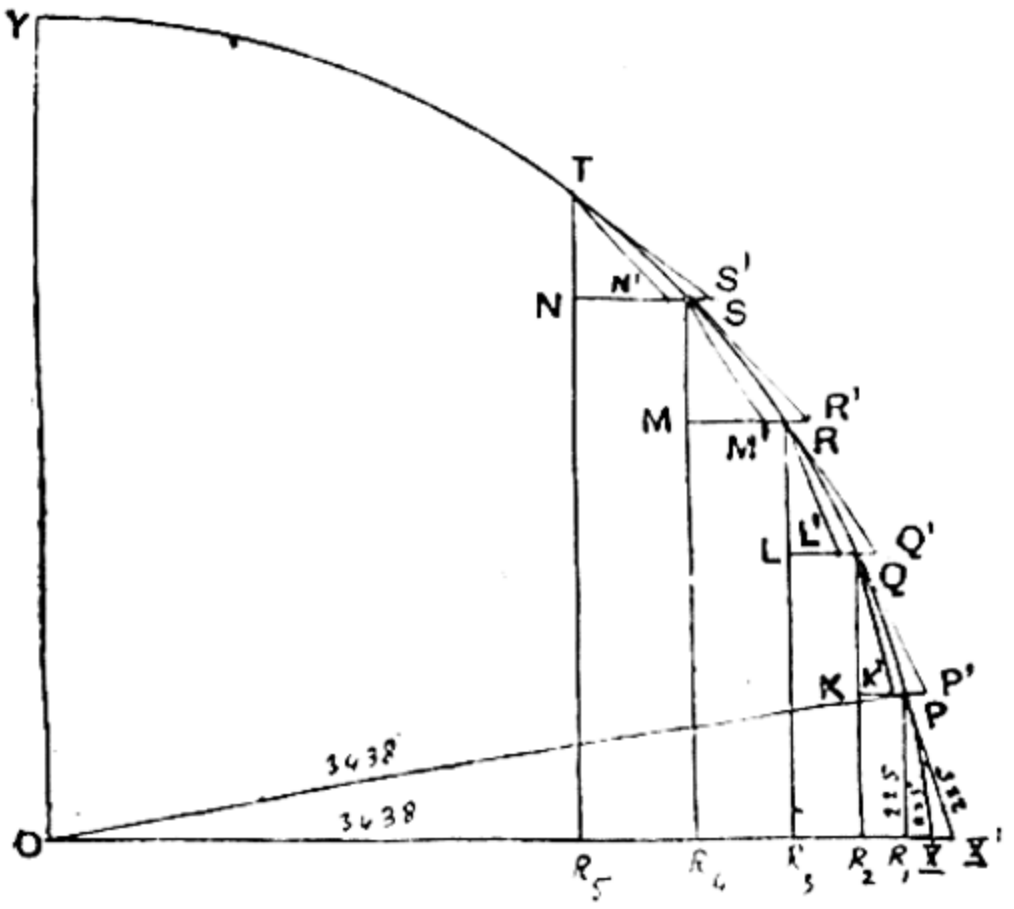
$$Q_2 = \frac{S_2}{225} = \frac{449}{225} = 2 ; \therefore S_3 = S_2 + (225 - \overline{1 + 2})$$
$$= 449 + 222 = 671.$$

$$Q_3 = \frac{S_3}{225} = \frac{671}{225} = 3 ; \therefore S_4 = S_3 + (225 - \overline{1 + 2 + 3})$$
$$= 671 + 219 = 890.$$

Thus Q_r becomes 1, 2, 3, etc., for different values of r in order and therefore ΣQ_r is the sum of certain natural numbers in a progressive series (which the Hindus called श्रेढी गणित) and it is possibly this that could have suggested to them the formulation of the above rule subject of course to the corrections to be applied.

Now as already considered, the sine, the arc and the tangent corresponding to 225′ present no appreciable differences and therefore if, in the following figure, OXY is a quadrant of a circle, XP, PQ, QR, etc., are divisions ot the circumference into arcs of 225′ each, PX, QP, RQ, etc., can be treated as tangents to the circle at P, Q, R, etc., and if prependiculars PR_1, QR_2, RR_3, etc., are drawn from P_1, Q_1, R_1, etc., to OX, then they are in order the first, second, third, etc., sines.

Now though the first sine PR_1 and the arc PX measure 225 each, they do not coincide and between them there is the



$$\text{versed sine } XR_1 = OX - OR_1 = OX - \sqrt{OP^2 - PR_1^2}$$
$$= 3438 - \sqrt{3438^2 - 225^2}$$
$$= 3438 - 3431 \text{ (clearing of fraction)}$$
$$= 7.$$

Similarly the tangent at P, though practically equal to PX, does not coincide with it, but takes some such position as PX' in the figure. So also the tangents at Q, R, S, etc., take positions as QP', RQ', SR', etc.

Now if PK, QL, RM, etc., are perpendiculars to QR_2, RR_3, SR_4, etc., from P, Q, R, etc., then QK, RL, SM, etc., are successive differences to be added to the preceding sines to give the following sines.

Draw also QK', RL', SM', etc., parallel to PX', QP', RQ', etc.

Now coming to approximations (which our old writers often do and also deduce values of unknown quantities by comparing similar triangles), R_1PX and $X'PX$ could be treated as triangles equal to one another in all respects.

$$\therefore R\hat{P}_1X = X'\hat{P}'X$$

whence $$R\hat{P}X = 2.225', \text{ and } R_1X = XX' = 7.$$

Again approximately, QK', QP' and QP' correspond to PR_1, PX, and PX', and the triangle QKK' is equal to the triangle PR_1X' in all respects;

$$\therefore \quad KK' = R_1X' = 14,$$

and $$K'P = PP' = 7.$$

$$\therefore \quad KP = KK' + K'P$$
$$= 14 + 7$$
$$= 21,$$

whence from the right-angled triangle QKP

$$QK = \sqrt{PQ^2 - KP^2}$$
$$= \sqrt{225^2 - 21^2}$$
$$= 224 \text{ (clearing of fraction).}$$

Similarly $$LQ = 5 \times 7 = 35;$$

$$\therefore \quad RL = \sqrt{RQ^2 - LQ^2}$$
$$= \sqrt{225^2 - 35^2}$$
$$= 222 \text{ (clearing of fraction)}$$

and so on.

Now with the same notation as before,

$$D_1 = 224 = 225 - 1$$
$$= S_1 - Q_1 \quad (\because Q_1 = 1);$$

$$D_2 = 222 = 225 - \overline{1 + 2}$$
$$= S_1 - \overline{Q_1 + Q_2} \quad (\because Q_2 = 2);$$

and generally,

$$D_{r-1} = 225 - [1 + 2 + 3 + \ldots\ldots + (r - 1)]$$
$$= S_1 - \Sigma Q_{r-1}$$

i.e., $$S_r - S_{r-1} = S_1 - \Sigma Q_{r-1} \quad (\because S_r - S_{r-1} = D_{r-1})$$

and also

$$D_{r-1} - D_r = \Sigma Q_r - \Sigma Q_{r-1} = Q_r,$$

results which prove the rule from either aspect.

It could be seen that $Q_1 = 1$, and for all other values of r, Q involves a fraction, and is reduced to an integer by the rule for clearing of fractions, *viz.*—

(i) *A general convention* to drop the fractional part of a number and retain only the integral part of it when the fractional part is less than $\frac{1}{2}$; and in all other cases, the fractional part is dropped and the integral part is increased by unity (अर्धाधिकावयवस्यैकाधिकत्वेन ग्रहस्य सांप्रदायिकत्वात्— Ranganatha.)

n rigidly following the above rule certain excesses might continually accumulate or diminish and in either case the accuracy of the results is interfered with. Therefore to keep them within the range of accuracy the following two rules are adopted :—

(ii) A *special convention* which has to be observed only in calculating this table of 24 sines is this:—

ऐकविंशाच्च विंशाच्च षष्ठात्पंचदशादपि ।
सप्तमाद्द्वादशात्सप्तदशान्नार्धोत्तरं मतम् ॥

(*Brahma-Siddhanta* cited by Ranganatha)

which in effect means—the fractional part though not less than $\frac{1}{2}$ should be dropped and the integral part only retained in Q_6, Q_7, Q_{12}, Q_{15}, Q_{17}, Q_{20}, and Q_{21}.

(iii) Certain other *corrections* there are which could in effect be stated thus :

Diminish ΣQ_r by 1, 2, 3, 4, and 5 in order, before subtracting from S_1 when $r = 7, 8, 9, 10$ and 11; 7, 9 and 11, when $r = 12, 13$, and 14; 14, 17, and 20; when $r = 15$, 16, and 17; 25 and 30. when $r = 18$ and 19; 36, when $r = 20$; 43, when $r = 21$; 50, when $r = 22$; and 58, when $r = 23$.

Thus a complete general rule for calculating the 24 sines is laid down and it also satisfies the rule of proportional parts within those limits.

Example. To calculate $\sin 62^\circ\ 28'$.

Since $62^\circ 28' = 62 \times 60 + 28 = 3478'$,

$$\therefore \qquad \frac{3478}{225} = 16\frac{148}{225},$$

and the required sine lies between the 16th (= 2978) and 17th (=3084) sines, the difference between which is 3084 — 2978 = 106, which corresponds to 225′ ;

∴ diff. corresponding to 148 is

$$\frac{148}{225} \times 106 = 69\frac{163}{225}$$

i.e., $= 70$ (clearing of fraction)

∴ the required sine $= 2978 + 70$

$= 3048$

It should be observed that by dividing any one of the sines thus obtained by 3438 (or by 3437'74677 for greater accuracy), the values of Natural Sines as given in any modern book of Tables is obtained, *e.g.*, dividing 3048 above obtained by 3437'74677, we get

$$\frac{3048}{3437'74677} = '8866270 \quad \ldots \quad (1)$$

and from Chamber's *Tables* sin 62° 28′ = '8867420, which is practically the same as (1).

The above facts are based on the texts and the conventions of the Sûrya-Siddhânta. If any of the values are differently given by other writers it is not only because the value of π is differently taken, [*e.g.*, $\sqrt{10}$, 600/191 etc.], but also the part of the circumference which presents no curvature is differently assumed,

For example, it is only[6] the 110th part of the circumference that looks flat according to Aryabhatta, and so on. But it is hard to account for the values Brahmagupta gives, for though he apparently takes $\pi = \sqrt{10}$ and therefore his value of the radius ought to have been $\frac{21600}{2 \times \sqrt{10}} = 3430$, yet he gives it as 3270 and the first sine as 214. The proportions however hold good throughout: *e.g.*, if instead of 3438, 3270 is taken, then instead of 225, we have to take

$$\frac{225 \times 3270}{3438} = \frac{25 \times 3270}{382} = \frac{40875}{191} = 214\frac{1}{191} \; i.e., \; 214,$$

which is Brahmagupta's figure. It might be that, since he says राश्यष्टांशे ष्वङ्कान्, (XXI . 17) and not राशिलिप्ताष्टमोभागः etc., he might have chosen arbitrary figures to fix the relation between the sines and the radius; or, it is suggested with much diffidence, that he might have chosen the quadrant of an ellipse (the form of the orbit of any planet) whose minor axis is 3270, though its major axis might be 3430 or whatever it is[7].

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[6] पपनोभागः परिधेः (प = 1, प = 1, न = 0).

[7] The late Pandit Sudhakara Dvivedi in the course of his commentary on Brahmagupta (II. 6—9) says. आचार्येण किमर्थं त्रिज्याख मुनिरद (3270) मिता गृहीतेत्यस्य मीमांसा भूमिकायां भविष्यति । Certainly it would have been interesting to have the view of so high an exponent of Indian Astronomy as the late Pandit, and it is disappointing that his introduction contains nothing on the subject.

Lastly, the rule laid down by Aryabhatta thus—

प्रथमाच्चापज्यार्धाद्यैरूनं खंडितं द्वितीयार्धं ।
तत्प्रथमज्यार्धांशै स्त्यै स्त्यै रूनानि शेषाणि ॥

means exactly the same as the rule of the Surya Siddhanta quoted above, if it is interpreted as under : प्रथमात् चापज्यार्धात् खण्डितं भक्तं—प्रथममिति शेषः यैः भागलब्धांशैः ऊनं हीनं प्रथम मित्येव शेषः—(तैः सह) तत् प्रथमज्यार्धं द्वितीयार्धं द्वितीयज्यार्धं भवति—प्रथमज्यार्धांशैः तैः तथैव प्रथमज्यार्धविभक्तफलांशैः ऊनानि यानि फलानि तैः फलैः सह इति यावत्, शेषाणि इतराणि ज्यार्धानि स्युः ॥— that is, the first sine being divided by itself and the quotient obtained, the same (*i. e.*, the first sine) together with the same (*i.e.*, the first sine) *minus* the quotient is the second sine ; and the other sines are obtained by successively subtracting the sum of all the quotients from the first sine and adding the results successively to the last of the already obtained sines. Symbolically,

$$S_r = S_{r-1} + S_1 - \Sigma Q_{r-1}$$

which has already been proved.

S. N. NARAHARAYYA.

Solutions.

Question 359.

(S. RAMANUJAN): If $\sin(x+y) = 2\sin\frac{1}{2}(x-y)$ and $\sin(y+z) = 2\sin\frac{1}{2}(y-z)$, prove that

$$\sqrt[4]{(\tfrac{1}{2}\sin x\cos z)} + \sqrt[4]{(\tfrac{1}{2}\cos x\sin z)} = \sqrt[12]{(\sin 2y)}$$

and verify the result when

$$\sin 2x = (\sqrt{5}-2)^3(4+\sqrt{15})^2;\ \sin 2y = \sqrt{5}-2;$$
$$\sin 2z = (\sqrt{5}-2)^3(4-\sqrt{15})^2.$$

Solution by T. R. Srinivasa Iyer, M.A.

Now

$$\cos^4\frac{x+y}{2} - \sin^4\frac{x+y}{2} = \left(\cos^2\frac{x+y}{2} + \sin^2\frac{x+y}{2}\right)$$
$$\left(\cos^2\frac{x+y}{2} - \sin^2\frac{x+y}{2}\right) = \cos(x+y), \text{ identically} \quad \ldots (1)$$

and $$\cos^4\frac{x+y}{2} + \sin^4\frac{x+y}{2} = \left(\cos^2\frac{x+y}{2} + \sin^2\frac{x+y}{2}\right)^2$$
$$- 2\sin^2\frac{x+y}{2}\cos^2\frac{x+y}{2}$$
$$= 1 - \frac{\sin^2(x+y)}{2}$$
$$= 1 - 2\sin^2\frac{x-y}{2} \text{ (by hyp.)}$$
$$= \cos(x-y). \quad \ldots \quad \ldots (2)$$

From (1) and (2), we get

$$\cos^4\frac{x+y}{2} = \cos x\cos y, \quad \ldots \quad \ldots (3)$$

$$\sin^4\frac{x+y}{2} = \sin x\sin y. \quad \ldots \quad \ldots (4)$$

Similarly, since $\sin(y+z) = 2\sin\frac{y-z}{2}$, we get

$$\cos^4\frac{y+z}{2} = \cos y\cos z, \quad \ldots \quad \ldots (5)$$

$$\sin^4\frac{y+z}{2} = \sin y\sin z, \quad \ldots \quad \ldots (6)$$

From (4) and (5),

$$\mathscr{S}(\sin y \cos y \sin z \cos z) = \sin \frac{x+y}{2} \sin \frac{y+z}{2},$$

and from (3) and (6)

$$\mathscr{S}(\sin y \cos y \sin x \cos x) = \cos \frac{x+y}{2} \sin \frac{y+z}{2}.$$

$$\therefore \quad \mathscr{S} \sin 2y \left[\mathscr{S}(\tfrac{1}{2} \sin x \cos z) + \mathscr{S}(\tfrac{1}{2} \sin z \cos x) \right] = \sin \left(\frac{x+y}{2} + \frac{y+z}{2} \right). \quad \ldots (7)$$

It will now be shown that

$$\sin \left(\frac{x+y}{2} + \frac{y+z}{2} \right) = (\sin 2y)^{\frac{1}{3}}.$$

The given conditions may be written

$$2 \sin \frac{x+y}{2} \cos \frac{x+y}{2} = 2 \sin \left(\frac{x+y}{2} - y \right)$$

and

$$2 \sin \frac{y+z}{2} \cos \frac{y+z}{2} = 2 \sin \left(y - \frac{y+z}{2} \right).$$

Putting $\frac{x+y}{2} = \alpha$ and $\frac{y+z}{2} = \beta$, the given conditions are

$$\sin(\alpha - y) = \sin \alpha \cos \alpha \quad \ldots \quad \ldots (8)$$

and

$$\sin(y - \beta) = \sin \beta \cos \beta. \quad \ldots \quad \ldots (9)$$

$$\therefore \quad \sin \alpha \cos y - \cos \alpha \sin y = \sin \alpha \cos \alpha,$$

$$-\sin \beta \cos y + \cos \beta \sin y = \sin \beta \cos \beta.$$

Solving for $\sin y$ and $\cos y$,

$$\sin y (\cot \beta - \cot \alpha) = \cos \beta + \cos \alpha,$$

and

$$\cos y (\tan \alpha - \tan \beta) = \sin \alpha + \sin \beta.$$

$$\therefore \quad \sin y = \frac{\cos \beta + \cos \alpha}{\cot \beta - \cot \alpha} = \frac{2 \cos \frac{1}{2}(\alpha+\beta) \cos \frac{1}{2}(\alpha-\beta) . \sin \alpha \sin \beta}{\sin(\alpha - \beta)},$$

or

$$\sin y = \frac{\cos \frac{1}{2}(\alpha+\beta) . \sin \alpha \sin \beta}{\sin \frac{1}{2}(\alpha-\beta)}. \quad \ldots \quad (10)$$

Also $\cos y = \dfrac{\sin \alpha + \sin \beta}{\tan \alpha - \tan \beta} = \dfrac{2 \sin \frac{1}{2}(\alpha+\beta) \cos \frac{1}{2}(\alpha-\beta) \cos \alpha \cos \beta}{\sin(\alpha - \beta)}$

or

$$\cos y = \frac{\sin \frac{1}{2}(\alpha+\beta) \cos \alpha \cos \beta}{\sin \frac{1}{2}(\alpha-\beta)}. \quad \ldots \quad \ldots (11)$$

From (10) and (11), we get

$$\sin^2\tfrac{1}{2}(\alpha+\beta)\cos^2\alpha\cos^2\beta+\cos^2\tfrac{1}{2}(\alpha+\beta)\sin^2\alpha\sin^2\beta=\sin^2\tfrac{1}{2}(\alpha-\beta)$$

i.e. $\sin^2\tfrac{1}{2}(\alpha+\beta)\{\cos(\alpha+\beta)+\cos(\alpha-\beta)\}^2+\cos^2\tfrac{1}{2}(\alpha+\beta)$

$$\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\}^2=4\sin^2\tfrac{1}{2}(\alpha-\beta),$$

i.e. $$\{\cos^2(\alpha+\beta)+\cos^2(\alpha-\beta)\}\left(\cos^2\frac{\alpha+\beta}{2}+\sin^2\frac{\alpha+\beta}{2}\right)$$

$$-2\cos(\alpha+\beta)\cos(\alpha-\beta)\left(\cos^2\frac{\alpha+\beta}{2}-\sin^2\frac{\alpha+\beta}{2}\right)$$

$$=4\sin^2\frac{\alpha-\beta}{2}.$$

i.e. $$\cos^2(\alpha+\beta)+\cos^2(\alpha-\beta)-2\cos^2(\alpha+\beta)\cos(\alpha-\beta)$$

$$=4\sin^2\frac{\alpha-\beta}{2}.$$

$\therefore$ $$\cos^2(\alpha+\beta)+\cos^2(\alpha-\beta)$$

$$-2\cos^2(\alpha+\beta)\left\{1-2\sin^2\frac{\alpha-\beta}{2}\right\}=4\sin^2\frac{\alpha-\beta}{2}$$

i.e. $$\cos(\alpha-\beta)+\cos^2(\alpha+\beta)=4\sin^2\frac{\alpha-\beta}{2}.\sin^2(\alpha+\beta)$$

i.e. $$\{\cos(\alpha-\beta)+\cos(\alpha+\beta)\}\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\}$$

$$=4\sin^2\frac{\alpha-\beta}{2}\sin^2(\alpha+\beta)$$

i.e. $$\cos\alpha\cos\beta\,.\,\sin\alpha\sin\beta=\sin^2\frac{\alpha-\beta}{2}\sin^2(\alpha+\beta).$$

$\therefore$ $$\sin^2(\alpha+\beta)=\frac{\cos\alpha\cos\beta}{\sin\frac{1}{2}(\alpha-\beta)}\cdot\frac{\sin\alpha\sin\beta}{\sin\frac{1}{2}(\alpha-\beta)}$$

$$=\frac{\cos y}{\sin\frac{1}{2}(\alpha+\beta)}\cdot\frac{\sin y}{\cos\frac{1}{2}(\alpha+\beta)}\quad\text{from (10) and (11)}$$

$\therefore$ $$\sin^2(\alpha+\beta)=\frac{\sin 2y}{\sin(\alpha+\beta)}$$

i.e. $$\sin^3(\alpha+\beta)=\sin 2y$$

i.e. $$\sin(\alpha+\beta)=(\sin 2y)^{\frac{1}{3}}.$$

Replacing α and β, by $\frac{x+y}{2}$ and $\frac{y+z}{2}$, we have

$$\sin\left(\frac{x+y}{2}+\frac{y+z}{2}\right)=\sin(2y)^{\frac{1}{3}}. \quad \ldots \quad \ldots \quad (12)$$

But from (7), we have

$$(\sin 2y)^{\frac{1}{4}} \{ (\tfrac{1}{2} \sin x \cos z)^{\frac{1}{4}} + (\tfrac{1}{2} \sin z \cos x)^{\frac{1}{4}} \} = \sin \left[\frac{x+y}{2} + \frac{y+z}{2} \right].$$

∴ From (7) and (12), we get

$$(\sin 2y)^{\frac{1}{4}} \{ (\tfrac{1}{2} \sin x \cos z)^{\frac{1}{4}} + (\tfrac{1}{2} \sin z \cos x)^{\frac{1}{4}} \} = (\sin 2y)^{\frac{1}{3}}$$

∴ $(\tfrac{1}{2} \sin x \cos z)^{\frac{1}{4}} + (\tfrac{1}{2} \sin z \cos x)^{\frac{1}{4}} = (\sin 2y)^{\frac{1}{12}}$,

i.e. $\sqrt[4]{(\tfrac{1}{2} \sin x \cos z)} + \sqrt[4]{(\tfrac{1}{2} \sin z \cos x)} = \sqrt[12]{\sin 2y}$,

which is the result to be proved.

The numerical values given have been verified and found correct.

N.B.—Results (3), (4) and (12) were communicated to me by Mr. M. Bhimasena Rao, who obtained them by the use of Elliptic Functions.

Question 1226.

(B. B. BAGI):—ABC, DEF are six points on a conic, DE, DF meet BC in L_1, L_2; EF, ED meet CA in M_1, M_2; and FD, FE meet AB in N_1, N_2; then if AD, BE, CF meet in a point O, then L_1N_2, N_1M_2, M_1L_2 meet in the same point O.

Solution by P. R. Venkatakrishna Iyer.

Let O be taken as the origin and OCF and OBE be the axes of co-ordinates. If OC = c, OB = b, OF = f, OE = e, then the equation of any conic through B, C, E, F is

$$\left(\frac{x}{c} + \frac{y}{b} - 1\right)\left(\frac{x}{f} + \frac{y}{e} - 1\right) = \lambda xy. \qquad \ldots \quad (i)$$

Let A be (x_1, mx_1) and D be (x_2, mx_2), where x_1 and x_2 are the roots of the quadratic

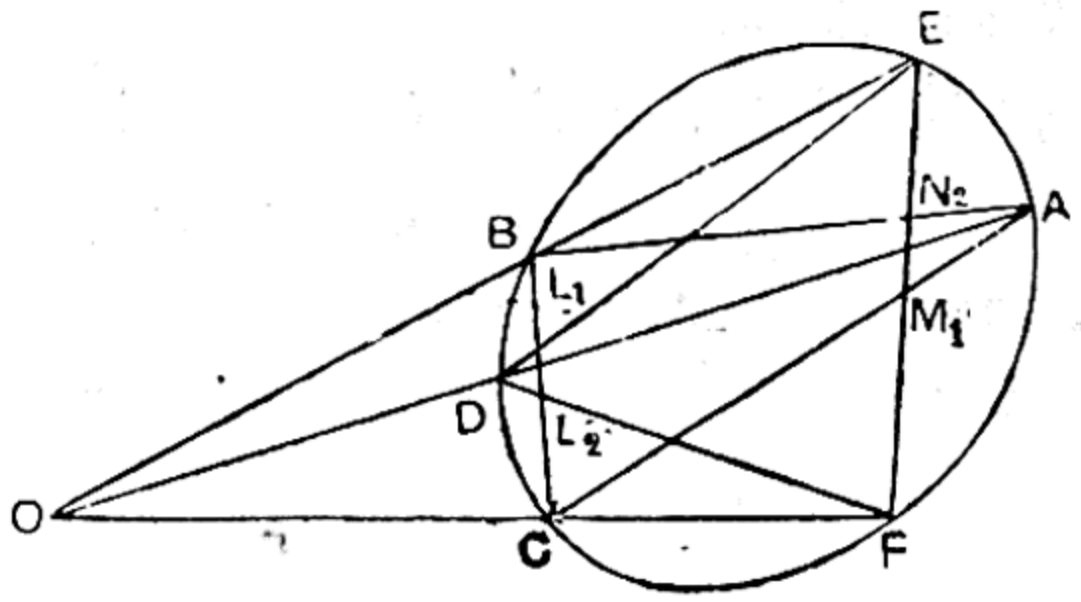
$$\left(\frac{x}{c} + \frac{mx}{b} - 1\right)\left(\frac{x}{f} + \frac{mx}{e} - 1\right) = \lambda m x^2,$$

i.e., of the equation

$$x^2 \left\{ \left(\frac{1}{c} + \frac{m}{b}\right)\left(\frac{1}{f} + \frac{m}{e}\right) - \lambda m \right\} - x \left(\frac{1}{c} + \frac{m}{b} + \frac{1}{f} + \frac{m}{e}\right) + 1 = 0. \quad (ii)$$

The straight lines DE and BC meet in L_1, whose co-ordinates (x, y) are given by the equations

$$y - e = \frac{x}{x_2}(mx_2 - e) \text{ and } \frac{x}{c} + \frac{y}{b} = 1.$$



From these two equations, we get

$$\frac{y}{x} = \frac{bx_2(e + mc) - bce}{(b - e)cx_2}.$$

Similarly, if N_2 is (x', y'), we get

$$\frac{y'}{x'} = \frac{ex_1(b + mf) - bef}{(e - b)fx_1}.$$

If L_1 and N_2 are in the same straight line with O, then

$$\frac{y}{x} = \frac{y'}{x'}; \text{ that is, } \frac{bx_2(e + mc) - bce}{(b - e)cx_2} = \frac{ex_1(b + mf) - bcf}{(e - b)fx_1}.$$

i.e.,
$$bcef\left(\frac{1}{x_1} + \frac{1}{x_2}\right) = bf(e + mc) + ce(b + mf),$$

i.e.,
$$\frac{1}{x_1} + \frac{1}{x_2} = \frac{1}{c} + \frac{m}{e} + \frac{1}{f} + \frac{m}{b},$$

which is true as x_1 and x_2 are the roots of the equation (ii).

Therefore, L_1N_2 passes through O.

Similarly N_1M_2 and M_1L_2 pass through O.

Question 1228.

(Prof. K. J. SANJANA) :—Given the intrinsic equation of a curve show how to find the intrinsic equations of its first positive and negative pedals.

Solution by S. Audinarayanan and K. Satyanarayana.

Suppose that the intrinsic equation of the curve is

$$s = f(\psi).$$

This can be transformed into tangential-polar equation thus :

$$\frac{d^2p}{d\psi^2} + p = f'(\psi).$$

Solving this, we have

$$p = A\sin\psi + B\cos\psi + \int_0^{\psi} f'(\omega)\sin(\psi - \omega)\,d\omega', \quad \ldots \quad (1)$$

where A and B are constants depending on the origin.—(Edward's *Integral Calculus*, Vol. I, p. 557.)

It is evident that the tangential-polar equation of a curve is the polar equation of its first positive pedal. Hence the polar equation of the first positive pedal is given by (1).

Let us suppose that (1) is represented by $r = \phi(\theta)$.

When this equation is transformed into an intrinsic equation, we have

$$s = \int \sqrt{\{\phi(\theta)\}^2 + \{\psi'(\theta)\}^2}\,d\theta.$$

(*Ibid*, p. 552). $\ldots$ (2)

Thus we can get the intrinsic equation of the first positive pedal.

When we come to the negative pedal, we have to reverse the process.

From the intrinsic equation of the curve

$$s = f(\psi)$$

get the polar equation of the curve by eliminating x, y and ψ from the equations

$$x = \int \cos\psi\, f'(\psi)\,d\psi$$
$$y = \int \sin\psi f'(\psi)\,d\psi$$
$$r = \sqrt{(x^2 + y^2)}$$
$$\theta = \tan^{-1}\frac{y}{x}.$$

Having obtained the polar equation which is the tangential polar equation of the first negative pedal, we can transform it into an intrinsic equation.

If the first be $p = \phi(\psi)$, then $s = \phi'(\psi) + \int \phi(\psi)\,d\psi$.

Questions for Solution.

1320. (V. V. SATYANARAYANA MURTI):—P, Q are points on the sides AB, AC of a triangle ABC such that the circle APQ touches BC at its middle point M. N is the middle point of PQ. R is a point on the circle so that RM = MN. RN produced cuts the circle at K. The tangent at K cuts CA produced at L. Show that AB divides CL and MK in the same ratio.

1321. (R. GOPALASWAMI):—If two points be the isogonal conjugates of a pair of polar conjugate points of a triangle, the pole of their joining line with respect to the circum-conic passing through them is the symmedian point of the triangle.

Deduce that the polar of the symmedian point with respect to the rectangular hyperbola of the circum-centroidal pencil is the Euler line.

1322. (R. GOPALASWAMI):—O is the circum-centre of a triangle ABC and AO, BO, CO meet the opposite sides at L, M, N. P, Q, R are points on the sides, so that DP = DL, EQ = EM, FR = FN, where DEF is the middle-point-triangle of ABC. Show that the ellipse which touches the sides of ABC at P, Q and R also touches the nine-points circle.

1323. (B. B. BAGI):—Two circles intersect in A, B. A straight line through A meets them again in P, Q. Show that the circum-circle of the triangle PBQ envelopes a cardioid.

1324. (K. J. SANJANA, M.A.):—Find the values of λ for which the conics $x^2+y^2-2xy\cos\omega=\lambda^3$ and $xy=kx+hy$ have single contact,—ω being the angle between the axes of co-ordinates and (h, k) being a fixed point. If in this case P be the point of contact, PM and PN its ordinate and abscissa, and D the foot of the perpendicular from the origin on MN, prove that the distance of D from M is equal to the distance of (h, k) from N.

1325. (S. M. SHAH, M.A.):—Sum the series

$$\sum_{n=0}^{\infty} b^n . e^{-\frac{(x+2na)^2}{c}}$$

where a, b, c are constants.

1326. (N. DURAI RAJAN):—If OA, OB, OC are three rods in a plane jointed at O, show how to determine the triangle ABC such that the radius of the circle ABC is a minimum.

Notes and Questions.

Asymptotic Expansion of Certain Series.

[The force at a point on the circumference of a circle having on it a number of equally spaced gravitating bodies of the same magnitude involves series of types

$$\Sigma \operatorname{csc}\left(a+\frac{m\pi}{n}\right) \text{ and } \Sigma \operatorname{csc}\left(a+\frac{m\pi}{n}\right)\cot\left(a+\frac{m\pi}{n}\right)$$

where $a>0$ and m takes all integral values from 0 to $n-1$; the force exerted on one of them by the others involves series of types

$$\Sigma \operatorname{csc}\frac{m\pi}{n} \text{ and } \Sigma \operatorname{csc}\left(\frac{m\pi}{n}\right)\cot\left(\frac{m\pi}{n}\right).$$

Various methods of summing these series for large values of n have been suggested*. The following method of summation may be new; it is also applicable to cases where some gravitating bodies are absent. The summation would then involve values of m from m_1 to m_2.]

1. A slightly generalised series may be taken and summed up:

If $0<\mathrm{R}\left\{a+\frac{m_1 r\pi}{n}\right\}<\mathrm{R}\left\{a+\frac{m_2 r\pi}{n}\right\}<\pi$, it is required to sum up

$$\sum_{m=m_1}^{m_2} \operatorname{csc}\left(a+\frac{mr\pi}{n}\right).$$

We start with the well-known summation formula† that: if $f(x)$ is bounded everywhere in the region

$$x_1<\mathrm{R}(x)<x_2$$

where x_1 and x_2 are two integers,

$$f(x_1)+f(x_1+1)+f(x_1+2)\ldots f(x_2)=\tfrac{1}{2}\{f(x_1)+f(x_2)\}+\int_{x_1}^{x_2} f(x)dx$$
$$+2\int_0^\infty \frac{\mathrm{I}[f(x_1-it)+f(x_2+it)]\,dt}{e^{2\pi t}-1},$$

where $\mathrm{R}(x)$ and $\mathrm{I}(x)$ denote the real and imaginary parts of x.

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* See G. A. Schott: *The Electromagnetic Theory of Radiation*, p. 216. G. A. Schott and G. N. Watson: *Quarterly Journal*, Vol. 47 (1916), pp. 311, *et seq*. A. Sommerfeld: *Ann. Der Physik* 53 (1917), p. 511.

† The form is slightly contracted from Whittaker and Watson: ***Modern Analysis***, 2nd edition: p. 145, Q. 7.

$$\text{Now} \int_{m_1}^{m_2} \operatorname{csc}\left(a + \frac{mr\pi}{n}\right) dm = \frac{n}{r\pi} \log\left\{\frac{\tan(a + m_1 r\pi/n)}{\tan(a + m_2 r\pi/n)}\right\}.$$

$$\mathrm{I}\left[\operatorname{csc}\left(a + \overline{m_1 - it}\,\frac{r\pi}{n}\right)\right] + \mathrm{I}\left[\operatorname{csc}\left(a + \overline{m_2 + it}\,\frac{r\pi}{n}\right)\right]$$

$$= \frac{\cos\left(a + \frac{m_1 r\pi}{n}\right)\sinh\frac{\pi rt}{n}}{\left[\cosh\frac{\pi rt}{n}\right]^2 - \left[\cos\left(a + \frac{m_1 r\pi}{n}\right)\right]^2}$$

$$- \frac{\cos\left(a + m_2\frac{r\pi}{n}\right)\sinh\frac{\pi rt}{n}}{\left[\cosh\frac{\pi rt}{n}\right]^2 - \left[\cos\left(a + m_2\frac{r\pi}{n}\right)\right]^2}$$

$$= \sum_{s=0}^{s-1} \sinh\frac{\pi rt}{n}\left[(\cos\theta_1)^{2s+1} - (\cos\theta_2)^{2s+1}\right]\left(\cosh\frac{\pi rt}{n}\right)^{-2s-2}$$

$$+ \frac{\sinh\frac{\pi rt}{n}\left(\cosh\frac{\pi rt}{n}\right)^{-2s}(\cos\theta_1)^{2s+1}}{\left[\cosh\frac{\pi rt}{n}\right]^2 - \left[\cos\theta_1\right]^2}$$

$$- \frac{\sinh\frac{\pi rt}{n}\cdot\left(\cosh\frac{\pi rt}{n}\right)^{-2s}\left(\cos\theta_2\right)^{2s+1}}{\left[\cosh\frac{\pi rt}{n}\right]^2 - \left[\cos\theta_2\right]^2},$$

where $$\theta_1 = a + m_1\frac{r\pi}{n}\,;\quad \theta_2 = a + m_2\frac{r\pi}{n}.$$

Hence it follows that

$$\sum_{m_1}^{m_2} \operatorname{csc}\left(a + mr\frac{\pi}{n}\right) = \left\{\frac{\sin\left(a + \overline{m_1 + m_2}\,\frac{r\pi}{2n}\right)\cos\left(\overline{m_1 - m_2}\,\frac{r\pi}{2n}\right)}{\cos(m_1 - m_2)\frac{r\pi}{n} - \cos\left(2a + \overline{m_1 + m_2}\,\frac{r\pi}{n}\right)}\right\}$$

$$+ \frac{n}{\pi r}\log\left[\frac{\tan\theta_2}{\tan\theta_1}\right] + 2\sum_{s=0}^{s-1} A_s\left[(\cos\theta_1)^{2s+1} - (\cos\theta_2)^{2s+1}\right]$$

$$+ R_s \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(I)}$$

where $$A_s = \int_0^\infty \frac{\sinh\frac{\pi rt}{n}\left[\cosh\frac{\pi rt}{n}\right]^{-2s-2}}{e^{2\pi t} - 1}\,dt:$$

and it is easily seen that

$$|\,R_s\,| \leqslant \left| \phi_s \left\{ \int_0^\infty \frac{\left(\sinh \frac{\pi r t}{n}\right)\left(\cosh \frac{\pi r t}{n}\right)^{-2s-2}}{e^{2\pi t}-1}\, dt \right\} \right.$$
$$\left. \left\{ (\cos\theta_1)^{2s+1} - (\cos\theta_2)^{2s+1} \right\} \right|$$

where $\phi_s \to 1$ as $s \to \infty$.

The evaluation of A_s is given below in § 3.

2. Again

$$\sum_{m_1}^{m_2} \operatorname{csc}\left(a + \frac{m\pi r}{n}\right) \cot\left(a + \frac{m\pi r}{n}\right)$$
$$= \tfrac{1}{2}\left[\csc\theta_1 \cot\theta_1 + \csc\theta_2 \cot\theta_2\right] + \frac{n}{\pi r}\left[\csc\theta_1 - \csc\theta_2\right]$$
$$+ 2\int_0^\infty \frac{\sinh\frac{\pi r t}{n}\,\sin\theta_1\left(\cos^2\theta_1 + \cosh^2\frac{\pi r t}{n}\right)}{\left(\cosh^2\frac{\pi r t}{n} - \cos^2\theta_1\right)^2}\, dt$$
$$- 2\int_0^\infty \frac{\sinh\frac{\pi r t}{n}\,\sin\theta_2\left[\cos^2\theta_2 + \cosh^2\frac{\pi r t}{n}\right]}{\left(\cosh^2\frac{\pi r t}{n} - \cos^2\theta_2\right)^2}\, dt$$
$$= \tfrac{1}{2}\left[\csc\theta_1 \cot\theta_1 + \csc\theta_2 \cot\theta_2\right] + \frac{n}{\pi r}\left[\csc\theta_1 + \csc\theta_2\right]$$
$$+ 2\sum_{s=0}(s+1)(\cos\theta_1)^{2s}\sin\theta_1\,(A_s + A_{s+1}\cos^2\theta_1)$$
$$- 2\sum_{s=0}(s+1)(\cos\theta_2)^{2s}\sin\theta_2\,(A_s + A_{s+1}\cos^2\theta_2) + R'_s. \quad \text{(II)}$$

The series (I) and (II) are seen to be convergent.

3. The next step would be to determine

$$\frac{1}{2\pi}\int_0^\infty \frac{\sin bt\,.(\cosh bt)^{-m}\,dt}{e^t - 1}, \text{ where } b = \frac{r}{2n}.$$

If
$$\operatorname{sech} bt = \sum_0^\infty (-)^v . E_{2v} . \frac{(bt)^{2v}}{2v!},$$

where $|\,bt\,| < \frac{\pi}{2}$,

then for all positive values of bt

$$\operatorname{sech} bt = \sum_{0}^{\mu-1} (-)^r E_{2v} \cdot \frac{(bt)^{2v}}{2v!} + \theta (-)^{\mu} E_{2\mu} \frac{(bt)^{2\mu}}{2\mu!},$$

where $0 < |\theta| < 1$*.

It can be similarly deduced by induction or otherwise that if

$$\sinh bt \, (\cosh bt.)^{-m} = \sum_{1}^{\infty} (-)^v . C_{2v} \frac{(bt)^{2v-1}}{(2v-1)!}, \text{ where } |bt| < \frac{\pi}{2},$$

it is always equal to

$$\sum_{1}^{\mu-1} (-)^v . C_{2v} . \frac{(bt)^{2v-1}}{(2v-1)!} + \theta (-)^{\mu} C_{2\mu} \frac{(bt)^{2\mu-1}}{(2r-1)!},$$

also that, if C_{2v} has the value given in this equation,

$$\sinh bt \, (\cosh bt)^{-m-2}$$

$$= \sum_{1}^{\mu-1} (-)^{v-1} \left[\frac{C_{2v+2} + (1-m)^2 C_{2r}}{m(m+1)} \right] \frac{(bt)^{2v-1}}{(2v-1)!} + R_{\mu},$$

where $R_{\mu} = \theta' (-)^{\mu-1} \left[\frac{C_{2\mu+2} + (1-m)^2 C_{2\mu}}{m(m+1)} \right] \frac{(bt)^{2\mu-1}}{(2\mu-1)!}$.

By this method if we know the co-efficients C_{2v} for a value $-m$, we can find for $-\overline{m+2}$ the new co-efficients and so on.

Knowing the value of C_{2v} for the value $-m$, substituting the series for $\sinh bt \, (\cosh bt)^{-m}$ in the integral

$$\int_{0}^{\infty} \frac{\sinh bt \, (\cosh bt)^{-m} \, dt}{e^t - 1},$$

we get it equal to

$$\sum_{1}^{\mu-1} (-)^{v-1} C_{2v} \zeta(2v) . b^{2v-1} + \theta (-)^{\mu-1} C_{2\mu} \zeta(2\mu) b^{2\mu-1},$$

$$0 < \theta < 1,$$

where $$\zeta(x) = \frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \ldots\ldots$$

The series for the integral is asymptotic for small values of b and a large number of gravitating bodies.

Hence $$A_s = \sum_{1}^{\mu-1} (-)^{v-1} C_{2v} (-2s-2) . \zeta(2v) b^{2v-1} + R_{\mu}$$

where $C_{2v}(-2s-2)$ means that C_{2v} is taken when $m = 2s+2$.

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* See G. N. Watson: *Quarterly Journal of Math.*, Vol. 47, pp. 302 et. seq.

Values of C_{2v} for $\sinh x\ (\cosh x)^{-m} = \Sigma\ C_{2v}\ (-)^v\ \dfrac{x^{2v-1}}{(2v-1)!}$.

v.	C_{2v}.
1	1
2	$3m-1$
3	$15m^2+1$
4	$21m(5m^2+5m+3)-1$
5	$445m^4+2520m^3+3150m^2+1322m+1$.

These values of C_{2v} are generally enough.

S. L. MALURKAR.

Cross-Ratio Invariants.

[A single infinity of points denoted by a variable y will be brought into (1, 1) corespondence with another denoted by a variable x by means of the homographic relation

$$F(y) \equiv y(a_1 + b_1x) + a_0 + b_0x = 0,$$

or what is the same thing by

$$\phi(x) \equiv x(b_0 + b_1y) + a_0 + a_1y = 0.$$

If x_1, x_2, x_3, x_4 be four points of the x-set, corresponding to y_1, y_2, y_3, y_4 of the y-set, then it is known that the cross-ratio

$$(x_1x_2x_3x_4) = (y_1y_2y_3y_4).$$

This note obtains a simple generalisation of this well-known result.]

1. Consider the equation

$$F(y) \equiv y^n(a_n+b_nx) + y^{n-1}(a_{n-1} + b_{n-1}x) + \ldots + y(a_1 + b_1x) + a_0 + b_0x = 0.$$

or, what is the same thing,

$$\Phi(x) \equiv x(b_0 + b_1y + \ldots\ldots + b_ny^n) + (a_0 + a\ y + \ldots\ldots + a_ny^n) = 0.$$

For any particular value of y, $\phi(x) = 0$ gives only one value of x; while for any particular value of x, the same relation $F(y) = 0$ gives n values of y. That is, corresponding to a single infinity of points of the x-set, we obtain a single infinity of groups of points of the y-set, each group containing n points.

2 In this correspondence between n-point groups of the y-set and single points of the x-set, let us suppose four single points x_1, x_2, x_3, x_4 correspond respectively to the four groups $(\alpha_1\alpha_2\ldots\alpha_n)$, $(\beta_1\beta_2\ldots\beta_n)$, $(\gamma_1\gamma_2\ldots\gamma_n)$ $(\delta_1\delta_2\ldots\delta_n)$.

We shall write

$$[\bar{\alpha}\ \beta\ \bar{\gamma}\ \delta]_n \equiv (\alpha_l\ \beta_1\ \gamma_k\ \delta_1)(\alpha_l\ \beta_2\ \gamma_k\ \delta_2)\ (\alpha_l\ \beta_3\ \gamma_k\ \delta_3)\ldots\ldots(\alpha_l\ \beta_n\ \gamma_k\ \delta_n)$$

and $[\alpha\ \bar{\beta}\ \gamma\ \bar{\delta}]_n \equiv (\alpha_1\ \beta_l\ \gamma_1\ \delta_k)\ (\alpha_2\ \beta_l\ \gamma_2\ \delta_k)\ (\alpha_3\ \beta_l\ \gamma_3\ \delta_k)\ldots\ldots(\alpha_n\ \beta_l\ \gamma_n\ \delta_k)$, l and k being any integers between 1 and n.

$[\bar{\alpha}\ \beta\ \bar{\gamma}\ \delta]_n$ and $[\alpha\ \bar{\beta}\ \gamma\ \bar{\delta}]_n$ may be called the *cross-ratio invariants* of the four n-point groups ; in the course of the following theorem, it will be evident that their values are independent of the integers l and k used in defining them.

The theorem of this note is

$$[\bar{\alpha}\ \beta\ \bar{\gamma}\ \delta]_n = [\alpha\ \bar{\beta}\ \gamma\ \bar{\delta}]_n = (x_1x_2x_3x_4).$$

Proof: By definition

$$[\bar{\alpha}\ \beta\ \bar{\gamma}\ \delta]_n = \frac{(\alpha_l - \beta_1)\ (\gamma_k - \delta_1)}{(\alpha_l - \delta_1)\ (\gamma_k - \beta_1)} \times \frac{(\alpha_l - \beta_2)\ (\gamma_k - \delta_2)}{(\alpha_l - \delta_2)\ (\gamma_k - \beta_2)} \times \ldots \times \frac{(\alpha_l - \beta_n)\ (\gamma_k - \delta_n)}{(\alpha_l - \delta_n)\ (\gamma_k - \beta_n)}$$

$$= \frac{(\alpha_l - \beta_1)\ (\alpha_l - \beta_2) \ldots (\alpha_l - \beta_n)}{(\alpha_l - \delta_1)\ (\alpha_l - \delta_2) \ldots (\alpha_l - \delta_n)} \times \frac{(\gamma_k - \delta_1)\ (\gamma_k - \delta_2) \ldots (\gamma_k - \delta_n)}{(\gamma_k - \beta_1)\ (\gamma_k - \beta_2) \ldots (\gamma_k - \beta_n)}.$$

Now $F(y) = 0$ has roots $\beta_1, \beta_2 \ldots \beta_n$ when $x = x_2$ and $\delta_1, \delta_2 \ldots \delta_n$ when $x = x_4$. So

$$[\bar{\alpha}\ \beta\ \bar{\gamma}\ \delta]_n = \frac{F(\alpha_l)_{x=x_2}}{F(\alpha_l)_{x=x_4}} \times \frac{F(\gamma_k)_{x=x_4}}{F(\gamma_k)_{x=x_2}}$$

$$= \frac{\phi(x_2)_{y=\alpha l} \times \phi(x_4)_{y=\gamma_k}}{\phi(x_4)_{y=\alpha l} \times \phi(x_2)_{y=\gamma_k}}.$$

Now $\phi(x) = 0$ has root x_1 when $y = \alpha_l$ and has root x_3 when $y = \gamma_k$ whatever l or k may be.

Hence

$$[\bar{\alpha}\ \beta\ \bar{\gamma}\ \delta]_n = \frac{(x_2 - x_1)}{(x_2 - x_3)} \times \frac{(x_4 - x_3)}{(x_4 - x_1)} = \frac{(x_1 - x_2)\,(x_3 - x_4)}{(x_1 - x_4)\,(x_3 - x_2)}$$

$$= (x_1\ x_2\ x_3\ x_4).$$

Similarly $[\alpha\ \bar{\beta}\ \gamma\ \bar{\delta}]_n = (x_1\ x_2\ x_3\ x_4)$.

Hence if a single infinity of points be in (1, 1) correspondence with a single infinity of groups of points each containing n-points, then the cross-ratio of any four points is equal to the *cross-ratio invariants* of the four corresponding n-point groups. As an illustration :

If four conics of a pencil be intersected by any fifth conic in the four tetrads of points, P_r, Q_r, R_r, S_r, $(r = 1, 2, 3, 4)$, then the product

$$(PQ_1RS_1)\,(PQ_2RS_2)\,(PQ_3RS_3)\,(PQ_4RS_4)$$

$$= (P_1QR_1S)\,(P_2QR_2S)\,(P_3QR_3S)\,(P_4QR_4S)$$

P, Q, R, S denoting any of the points P_r, Q_r, R_r, S_r ; also, the product is independent of the fifth conic, and equal to the cross-ratio of the four given conics.

R. GOPALASWAMI.

Solutions.

Question 714.

(K. B. MADHAVA) :—Find the co-efficient of x^n in

$$\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^4}{1+x^4} + \ldots\ldots$$

Solution by S. Audinarayanan.

On expanding each term and re-arranging in the form of a double series, it is easy to derive the co-efficient of x^n in the following manner:

Let $n = (\alpha_1 \times \beta_1)$, or $(\alpha_2 \times \beta_2) \ldots\ldots$ or $(\alpha_s \times \beta_s)$;

where α and β are the several conjugate factors of n and $\alpha_k = \beta_k$ say, when n is a perfect square. Then the co-efficient of x^n in the above series is $\Sigma(\lambda_r)$, where λ_r is

$+ 2$ when α and β are either both odd or both even,

$- 2$, when of α and β one is odd and the other even,

and $+ 1$, when $\alpha_k = \beta_k = \sqrt{n}$.

Question 771.

(K. B. MADHAVA) :—In his solution to Q. 629, Mr. Bhimasena Rao uses Bromwich, p. 447, Ex. 4, (after the method of Dirichlet's integrals), to show that if

$$\psi(t) = \sum_{-\infty}^{\infty} e^{-n^2\pi t}$$

we shall have

$$\psi(t) = t^{-\frac{1}{2}}\,\psi\left(\frac{1}{t}\right).$$

This result is due to Jacobi. See: *Ges. Werke.* II, p. 188. Verify this by integrating

$$\int \frac{e^{-z^2\pi t}}{e^{2\pi i z} - 1}\,dz$$

round a rectangle whose vertices are $\pm(R + \frac{1}{2}) \pm i$ and making $R \to \infty$ in the usual manner.

Show also by the same method, that if $t > 0$

$$\sum_{n=-\infty}^{\infty} e^{-n^2\pi t - 2n\pi t a} = t^{-\frac{1}{2}}\, e^{\pi a^2 t}\left\{1 + 2\sum_{1}^{\infty} e^{-\frac{n^2\pi}{t}} \cos 2n\pi a\right\},$$

also due to Jacobi.

Solution by S. Audinarayanan.

This is given in Whittaker and Watson: *Modern Analysis*, p. 124. We shall establish as a Lemma the result that if $\lambda > 0$

$$\int_{-\infty}^{\infty} e^{-\lambda x^2} \cos(2\lambda a x)dx = e^{-\lambda a^2} \int_{-\infty}^{\infty} e^{-\lambda x^2} dx$$

$$= \lambda^{-\frac{1}{2}} e^{-\lambda a^2} \int_{-\infty}^{\infty} e^{-x^2} dx.$$

(W. W., p. 114)

Integrate $\int e^{-\lambda z^2} dz$ round the rectangle $[-R, R, R+ai, -R+ai]$, and make $R \to \infty$.

Since the integrand is analytic inside the contour $\int e^{-\lambda z^2} dz = 0$. Hence we can write

$$\int_{-R}^{R} e^{-\lambda z^2} dz + \int_{R}^{R+ai} e^{-\lambda z^2}.dz + \int_{R+ai}^{-R+ai} e^{-\lambda z^2} dz + \int_{-R+ai}^{-R} e^{-\lambda z^2} dz = 0.$$

Put $z = R + iy$ in the second integral and it transforms into

$$\left| \int_0^a e^{-\lambda(R+iy)^2} i\, dy \right| = \left| i \int_0^a e^{-\lambda(R^2+y^2)-2R\lambda iy} dy. \right|$$

$$< \int_0^a e^{-\lambda(R^2-y^2)} dy$$

$$< a e^{-\lambda(R^2-a^2)} \to 0, \text{ as } R \to 0.$$

Similarly $\int_{-R+ai}^{-R} e^{-\lambda z^2} dz \to 0$, as $R \to \infty$.

The first integral is $\int_{-\infty}^{\infty} e^{-\lambda x^2} dx$.

Put $z = (x + ai)$ in the third. It becomes

$$\lim \int_{R}^{-R} e^{-\lambda(x+ai)^2} dx = e^{a^2\lambda} \int_{\infty}^{-\infty} e^{-\lambda x^2} . e^{-2aix\lambda} dx$$

$$= e^{a^2\lambda} \int_{\infty}^{-\infty} e^{-\lambda x^2} (\cos 2a\lambda x - i \sin 2a\lambda x)$$

Hence we have

$$\int_{-\infty}^{\infty} e^{-\lambda x^2} dx + e^{a^2\lambda} \int_{\infty}^{-\infty} e^{-\lambda x^2} (\cos 2a\lambda x - i \sin 2a\lambda x)\, dx = 0.$$

Equating real and imaginary parts

$$\int_{-\infty}^{\infty} e^{-\lambda x^2} \cos(2a\lambda x)\, dx = e^{-\lambda a^2} \int_{-\infty}^{\infty} e^{-\lambda x^2}\, dx$$

$$= \lambda^{-\frac{1}{2}} e^{-\lambda a^2} \int_{-\infty}^{\infty} e^{-x^2}\, dx$$

and $$\int_{-\infty}^{\infty} e^{-\lambda x^2} \sin(2a\lambda x)\, dx = 0.$$

Now we shall integrate $\int \frac{e^{-z^2\pi t}}{e^{2\pi tz} - 1}\, dz$ along the given contour, *i.e.*, the rectangle with vertices at $\pm (R + \frac{1}{2}) \pm i$.

The poles of the integrand inside the contour are at

$$\pm 1, \pm 2, \pm 3, \ldots\ldots \pm R.$$

The residue at $z = r$ is given by the co-efficient of $\frac{1}{s}$ on putting $z = r + s$ and expanding. The residue is

$$\frac{1}{2\pi i} e^{-2\pi t}.$$

Hence the sum of the residues at poles inside the contour

$$= \frac{1}{2\pi i} \sum_{-R}^{R} e^{-n^2\pi t}.$$

Therefore

$$\int_{-(R+\frac{1}{2})-i}^{(R+\frac{1}{2})-i} \frac{e^{-z^2\pi t}}{e^{2\pi tz} - 1}\, dz + \int_{(R+\frac{1}{2})-i}^{(R+\frac{1}{2})+i} \frac{e^{-z^2\pi t}}{e^{2\pi tz} - 1}\, dz + \int_{R+\frac{1}{2}+i}^{-(R+\frac{1}{2})+i} \frac{e^{-z^2\pi t}}{e^{2\pi tz} - 1}\, dz$$

$$+ \int_{-(R+\frac{1}{2})+i}^{-(R+\frac{1}{2})-i} \frac{e^{-z^2\pi t}}{e^{2\pi tz} - 1}\, dz = 2\pi i \frac{1}{2\pi i} \sum_{-R}^{R} e^{-n^2\pi t}.$$

When $R \to \infty$, the absolute values of the second and the fourth integrals tend to zero and we have

$$\int_{-\infty-i}^{\infty-i} \frac{e^{-z^2\pi t}}{e^{2\pi tz} - 1}\, dz + \int_{\infty+i}^{-\infty+i} \frac{e^{-z^2\pi t}}{e^{2\pi iz} - 1}\, dz$$

$$= \sum_{-\infty}^{\infty} e^{-n^2\pi t} = \psi(t).$$

Expanding the integrands in powers of $e^{-2\pi iz}$, $e^{2\pi iz}$ respectively, we have

$$\psi(t) = \int_{-\infty-i}^{\infty-i} e^{-z^2\pi t}\,[e^{-2\pi iz} + e^{-4\pi iz} + e^{-6\pi iz} + \ldots\ldots]\,dz$$

$$+ \int_{-\infty+i}^{\infty+i} e^{-z^2\pi t}\,(1 + e^{2\pi iz} + e^{4\pi iz} + e^{6\pi iz} + \ldots\ldots)\,dz.$$

Put $(x-i)$ and $(x+i)$ respectively for z in the two integrals, then we have,

$$\psi(t) = \int_{-\infty}^{\infty} \Big[e^{-(x^2+i)^2\pi t} + 2\sum_1^\infty \Big\{ e^{-x^2\pi t + \pi t - 2n\pi} \cos 2\pi t \left(1 - \frac{n}{t}\right)_x \Big\} \Big]\,dx.$$

By the Lemma we have

$$\psi(t) = \frac{e^{\pi t - \pi t}}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-x^2}\,dx + \sqrt{\frac{2}{\pi t}} \sum_1^\infty e^{\pi t - 2n\pi - \left(1-\frac{n}{t}\right)^2 \pi t} \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

$$= \frac{1}{\sqrt{\pi t}} \left[1 + 2\sum_1^\infty e^{-\frac{n^2}{t}\pi} \right] \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

$$= \frac{1}{\sqrt{\pi t}} \left\{ \sum_{-\infty}^{\infty} e^{-\frac{n^2\pi}{t}} \right\} \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

$$= \frac{1}{\sqrt{\pi t}} \cdot \psi\left(\frac{1}{t}\right) \int_{-\infty}^{\infty} e^{-x^2}\,dx.$$

But $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$, which can also be seen by putting $t = 1$.

$$\therefore \qquad \psi(t) = t^{-\frac{1}{2}}\,\psi\left(\frac{1}{t}\right).$$

The second part of the result comes out easily, if we have $z + a$ in place of z. Then

$$\sum_{-\infty}^{\infty} e^{-(z+a)^2\pi t} = \int_{-\infty-i}^{\infty-i} e^{-(z+a)^2\pi t}\,[e^{-2\pi i(z+a)} + e^{-4\pi i(z+a)} + \ldots]\,dz$$

$$+ \int_{-\infty+i}^{\infty+i} e^{-(z+a)^2\pi t}\,[1 + e^{2\pi i(z+a)} + e^{4\pi i(z+a)} + \ldots\ldots]\,dz$$

$$= \int_{-\infty}^{\infty} [e^{-(x+a+i)^2\pi t} + 2\sum_1^{\infty} \{ e^{-(x-i+a)^2\pi t - 2n\pi i(x+a-i)} + e^{-(x+a+i)^2 + 2n\pi i(x+a+i)} \}] \, dx$$

$$= \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-x^2} \, dx$$

$$+ 2\int_{-\infty}^{\infty} \sum_1^{\infty} e^{-(x+a)^2 + \pi t - 2n\pi} \cos 2\pi t \left(1 - \frac{n}{t}\right) x \cos 2n\pi a \, . \, dx$$

$$= \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-x^2} \, dx . \left[1 + 2\sum_1^{\infty} e^{-\frac{n^2\pi}{t}} \cos 2n\pi a\right]$$

by the Lemma

$$= t^{-\frac{1}{2}} [1 + 2\sum_1^{\infty} e^{-\frac{n^2\pi}{t}} \cos 2n\pi a] .$$

$$\therefore \quad \sum_1^{\infty} e^{-n^2\pi t - 2n\pi ta} = t^{-\frac{1}{2}} e^{a^2\pi t} (1 + 2\sum_1^{\infty} e^{-\frac{n^2\pi}{t}} \cos 2n\pi a).$$

Question 1197.

(N. B. MITRA) :—If $(2n + 1)$ is a prime number p, prove that the numerator of $2^{4n} + p\,(1 + \frac{1}{2} + \frac{1}{3} + \ldots + 1/n) - 1 \equiv 0 \pmod{p^2}$.

Solution by I. Totadri Iyengar.

$$(p-1)! = (p-1)\,(p-2)\,(p-3) \ldots (p-n) \, . \, n \, . \, (n-1) \ldots 2.1$$
$$= n! \, . \, (p-1)\,(p-2) \ldots\ldots (p-n)$$
$$\equiv n! \, . \, [(-1)^n] \; [n! - n!\,p\,(1 + \tfrac{1}{2} + \ldots + \tfrac{1}{n})] \bmod p^2$$

and hence $(p-1)! \equiv$ numerator of $(-1)^n\,(n!)^2$
$\times [1 - (1 + \frac{1}{2} + \ldots\ldots + 1/n)\,p]$, mod p^2.

But in my solution to Q. 1247 (see : Vol. XV, No. 4), it has been proved that

$$(p-1)! = 2n! \equiv (-1)^n\,2^{4n}\,(n!)^2, \bmod p^2.$$

$$\therefore \quad (-1)^n\,(n!)^2 \; [1 - (1 + \tfrac{1}{2} + \ldots\ldots + \tfrac{1}{n})\,p] \equiv (-1)^n\,2^{4n}(n!)^2, \bmod p^2$$

$$\therefore \quad 2^{4n} + (1 + \tfrac{1}{2} + \ldots\ldots + \tfrac{1}{n})\,p - 1 \equiv 0, \bmod p^2$$

on removing the common factors and taking the quantities to one side.

Question 1260.

(A. T. THOMAS):—If $a_1 = \sin x$, $a_2 = \sin \sin x$, and so on, where $0 < x < \pi$, prove that the infinite series Σa_n and Σa_n^2 are divergent but that Σa_n^3 is convergent; and, in fact, that if n be large $a_n \sim \sqrt{(3/n)}$.

Solution by A. Narasinga Rao.

We have here a sequence of numbers a_n decreasing steadily but never negative. Such a sequence necessarily tends to a limit which may be finite or zero. To determine this limit λ, we have to solve the equation $\sin \lambda = \lambda$. One root is $\lambda = 0$ and since λ increases more rapidly than $\sin \lambda$ there cannot be any other real root. Hence the sequence tends to zero and it remains to fix the order of a_n.

Since in the limit $$a_{n+1} = \sin a_n, \quad \ldots \quad \ldots (1)$$ it follows that

$$\log \frac{a_n}{a_{n+1}} = -\log\left(1 - \frac{a_n^2}{3!} + \frac{a_n^4}{5!} - \ldots\ldots\right) \quad \ldots (2)$$

$$= \frac{a_n^2}{6} + 0\,(a_n^4).$$

If a_n be of order n^{-r}, we have $\frac{a_n}{a_{n+1}} = \left(1 + \frac{1}{n}\right)^r$,

and so $$\log \frac{a_n}{a_{n+1}} = \frac{r}{n} + 0\left(\frac{1}{n^2}\right). \quad \ldots \quad \ldots (3)$$

Substituting in (2), we have the principal part of the first member of (2) equal to (r/n) while that of the second member is of order n^{-2r} and as these must be equal $r = \frac{1}{2}$. We may therefore take

$$a_n = \frac{\alpha}{\sqrt{n}} + \frac{\beta}{n} + 0\left(\frac{1}{n}\right). \quad \ldots \quad \ldots (4)$$

Substituting for a_n and a_{n+1} in

$$\frac{a_{n+1}}{a_n} = 1 - \frac{a_n^2}{2!} + \frac{a_n^4}{5!} \quad \ldots \quad \ldots (5)$$

we have

$$\frac{\frac{\alpha}{\sqrt{n+1}} + \frac{\beta}{n+1} + 0\left(\frac{1}{n}\right)}{\frac{\alpha}{\sqrt{n}} + \frac{\beta}{n} + 0\left(\frac{1}{n}\right)} = 1 - \frac{\alpha^2}{6n} - \frac{\alpha\beta}{3n^{\frac{3}{2}}} + 0\left(\frac{1}{n^2}\right). \quad \ldots (6)$$

The left side of (6) when expanded in ascending powers of $n^{-\frac{1}{2}}$ is found to be

$$1 - \frac{1}{2n} + 0\left(\frac{1}{n^{\frac{3}{2}}}\right)$$

and equating infinitesimals of the same order on both sides of (6), we have $\alpha = \sqrt{3}$.

If it is desired to carry the investigation further, we may take

$$a_n = \frac{\sqrt{3}}{\sqrt{n}} + \frac{\beta}{n} + \frac{\gamma}{n^{\frac{3}{2}}} + \ldots\ldots$$

substitute in (5) and equate the coefficients of different powers of $\frac{1}{\sqrt{n}}$ on both sides.

As $a_n \sim \sqrt{(3/n)}$, it is obvious that Σa_n and $\Sigma a_n{}^2$ diverge but that $\Sigma a_n{}^3$ converges.

Question 1261.

(A. T. THOMAS) :—If the numbers represented by $f(x, y)$, [where f is an integral expression with integral co-efficients of the positive integers x, y], are arranged in increasing order of magnitude and those less than N be considered, the probability that x and y are prime to each other is a number tending to $\frac{6}{\pi^2}$, as $N \to \infty$.

Solution by S. Audinarayanan.

Since f is an integral expression with integral co-efficients of positive integers x, y and since we consider numbers less than N, it is evident that the values of x and y are less than some number M. As N tends to infinity, M also tends to infinity. Hence the problem reduces itself to finding the probability of x and y being prime to each other, where x and y are less than M and $M \to \infty$.

Suppose that a prime number s is taken and x and y are not divisible by it. Then the probability of x and y being prime to each other is

$$P = p_2 \cdot p_3 \cdot p_5 \cdot \ldots p_s,$$

where the small p's are the probabilities that each of the primes 2, 3, 5, ... is not a common factor of x and y and s the highest prime $< M$.

Next, consider that x and y are divided by s. We are sure to get one of the following remainders 0, 1, 2, ... $(s - 1)$.

Now s divides x and y exactly where 0 is got as remainder. Thus the probability of s being a factor of x is $(1/s)$ and similarly for y. The probability of s being a common factor of x and y is $(1/s^2)$. So the probability of its not being a common factor is $\left(1 - \frac{1}{s^2}\right)$.

$$\therefore\ P = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\ldots\ldots\left(1 - \frac{1}{s^2}\right).$$

As M tends to infinity, P becomes an infinite product

Thus $$P = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right) \ldots\ldots \text{ to infinity}$$

$$\therefore \quad \frac{1}{P} = \frac{1}{1 - \frac{1}{2^2}} \cdot \frac{1}{1 - \frac{1}{3^2}} \cdot \frac{1}{1 - \frac{1}{5^2}} \ldots\ldots$$

$$= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots \text{ ad inf.} = \frac{\pi^2}{6}.$$

Hence $$P = \frac{6}{\pi^2}.$$

Question 1266.

(A. A. KRISHNASWAMI IYENGAR):—Find general expressions for the sides of rational triangles, the squares of whose areas are perfect cubes. (Ex. 5, 6, 7).

Solution by N. B. Mitra.

(i) If by a rational triangle we mean a triangle whose sides only are rational, then the triangle whose sides are $n^3(p^3 - pq^3)$, $n^3(p^2 - pq^3 + q^6)$ and $n^3(2pq^3 - q^6)$, where p, q and n are rational and p greater than q^3 will have its (area)^2 equal to a perfect cube, namely the cube of $n^4pq^2(p - q^3)$.

(ii) Ordinarily, however, by a rational triangle is meant a triangle whose sides as well as area are rational. In this case, the solution may be given thus: the triangle whose sides are

$n^3p^4(4p^2 - q^6)^2$; $n^3p^4(16p^4 + 56p^2q^6 + q^{12})$ and $16n^3p^5q^3(4p^3 + q^6)$ will have its area rational and a perfect cube, *viz.*, the cube of $2n^2p^3q(4p^2 - q^6)$ — p, q, n being rational.

Questions for Solution.

1327. (S. NARAYANA AIYAR, M.A.):—Show, from independent considerations, that the addition equations of the Jacobian elliptic functions may be thrown, in Dr. Glaisher's notation, into the forms

$$(1)\quad \tan^{-1}[\mathrm{sc}(u+v)] = \tan^{-1}\mathrm{sc}\,u.\,\mathrm{dn}\,v + \tan^{-1}\mathrm{sc}\,v.\,\mathrm{dn}\,u.$$

$$(2)\quad \tan^{-1}[k\,\mathrm{sd}(u+v)] = \tan^{-1}k\,\mathrm{sd}\,u.\,\mathrm{cn}\,v + \tan^{-1}k\,\mathrm{sd}\,v\,\mathrm{cn}\,u.$$

Hence, or otherwise, show that, if s_n denotes the sum of n quantities $u_1, u_2, u_3 \ldots\ldots u_n$, s_r the sum of any r of the quantities and s_{n-r} the sum of the remaining $n-r$ of the quantities

$$(3)\quad \mathrm{cn}\,s_n + i\,\mathrm{sn}\,s_n = \frac{\mathrm{cn}\,s_r + i\,\mathrm{sn}\,s_r.\,\mathrm{dn}\,s_{n-r}}{\mathrm{cn}\,s_{n-r} - i\,\mathrm{sn}\,s_{n-r}.\,\mathrm{dn}\,s_r}$$

$$(4)\quad \mathrm{dn}\,s_n + ik\,\mathrm{sn}\,s_n = \frac{\mathrm{dn}\,s_r + ik\,\mathrm{sn}\,s_r.\,\mathrm{cn}\,s_{n-r}}{\mathrm{dn}\,s_{n-r} - ik\,\mathrm{sn}\,s_{n-r}.\,\mathrm{cn}\,s_r}.$$

Examine the cases, when $\mathrm{sn}\,u = 1$ or $\frac{1}{k}$ in (1) and (2) above.

1328. (K. J. SANJANA and S. N. KUMARASWAMY):—On the major axis AA′ of an ellipse, whose minor axis is of length $2b$, a rectangle A′ADE is described, having the height $AD = b\sqrt{2}$; a straight line EG meets AA′ in G and cuts the ellipse at P and Q. If DP, DQ meet AA′, produced if necessary in F,F′, prove that A′F = A′F′ and that $(A'F'^2 + AG^2)$ is constant. When EG touches the ellipse at R, prove that DRA′ is a straight line.

1329. (A. A. KRISHNASWAMI IYENGAR, M.A.):—Prove that the greatest coefficient in the expansion of $(1 + 2x + 3x^2)^n$ is that of x^{4m}, x^{4m+2}, or x^{4m+3}, according as $n = 3m$, $3m+1$, or $3m+2$.

1330. (N. DURAIRAJAN):—If $I_1, I_2, I_3 \ldots\ldots$ be the centres of the inscribed circles of the triangles ABC, I_1BC, I_2BC....; show that I_n tends to a point I on BC, such that

$$\mathrm{BI} : \mathrm{IC} = \text{arc BA} : \text{arc AC}.$$

1331. (K. SATYANARAYANA):—Prove that the origin lies in the acute or obtuse angle between the straight lines

$$y = m_1 x + c_1,\ y = m_2 x + c_2,$$

according as

$$c_1 c_2 (1 + m_1 m_2) \lessgtr 0.$$

1332. (K. SATYANARAYANA) :—Prove that a triangle whose sides are parallel to $y = m_1x$, $y = m_2x$, $y = m_3x$, will be acute-angled, right-angled or obtuse-angled according as

$$(1 + m_1m_2)(1 + m_2m_3)(1 + m_3m_1) \gtreqless 0.$$

1333. (K. J. SANJANA) :—(X, X′), (Y, Y′), (Z, Z′) are points in the sides BC, CA, AB respectively of a triangle ABC, such that Y′Z, Z′X, X′Y are equal and respectively anti-parallel to BC, CA, AB. If circles are described about the triangles AY′Z, BZ′X, CX′Y, prove that the point isogonally conjugate to their radical centre lies on the *Euler line* of the triangle. Also state the corresponding property when the lines Y′Z, Z′ X, X′Y ot equal lengths are parallel respectively to the opposite sides of the triangle.

1334. (HEMRAJ) : –Triangles $X_1X_2X_3$, $X'_1X'_2X'_3$ [where $X_r = x_ry_rz_r$)] are self-conjugate for the conic $ux^2 + vy^2 + wz^2 = 0$. Conics through their vertices and those of the triangle of reference (two at a time) have a common self-conjugate triangle. Prove that

$$(1) \quad \prod_{r=1}^{r=3}(ux_r^2 + vy_r^2 + wz_r^2) \div \prod_{r=1}^{r=3}(ux_r'^2 + vy_r'^2 + wz_r'^2)$$

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}^2 \div \begin{vmatrix} x_1' & y_1' & z_1' \\ x_2' & y_2' & z_2' \\ x_3' & y_3' & z_3' \end{vmatrix}^2$$

$$(2) \quad \xi^2/u^3 + \eta^2/v^3 + \zeta^2/w^3 = 0,$$

where $\xi = y_1y_2y_3z'_1z'_2z'_3 - z_1z_2z_3y_1'y_2'y_3'$

$\eta = z_1z_2z_3x_1'x_2'x_3' - x_1x_2x_3z_1'z_2'z_3'$

and $\zeta = x_1x_2x_3y_1'y_2'y'_3 - y_1y_2y_3\ x_1'x_2'x_3'$.

1335. (MARTYN THOMAS) :—An endless inelastic string of uniform density is placed in a smooth tube in the form of a closed curve without singularities, and is acted on by a field of force such that unit mass experiences a force μr perpendicular to the radius vector. Show that the string will complete one revolution from rest in time $\frac{l}{\mu\sqrt{\Delta}}$, where l and Δ are the length and area of the curve.

Generalize the result when the force is μr^n, and examine the cases when $n = -1, -2$ and $\frac{3}{2}$.

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